Section 6.77. Behavior of Functions Near Isolated Singular Points

Note. For f analytic in $0 < |z - z_0| < R_2$, we describe the behavior of f for these z values when z_0 is one of each of the types of isolated singular points defined in Section 6.72.3, "The Three Types of Isolated Singular Points," namely removable singular points, poles, and essential singularities. First, we consider a pole.

Theorem 6.77.1. If z_0 is a pole of a function f then $\lim_{z \to z_0} f(z) = \infty$.

Note. In fact, the converse of Theorem 6.77.1 also holds, as can be seen by considering the Laurent series for f centered at z_0. So f has a pole at a_0 if and only if z_0 is an isolated singular point of f and $\lim_{z \to z_0} f(z) = \infty$. We could take this as the *definition* of "pole" and hence not have to appeal to Laurent series to define it. This is the approach taken by Conway in his graduate-level *Functions of One Complex Variable I*, Spring-Verlag (1978).

Note. Next, we consider the behavior of f in a deleted neighborhood of a removable singular point.

Theorem 6.77.2. If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

Note. We can use Theorem 6.77.2 to show that for f with removable singular point of z_0, $\lim_{z \to z_0}(z - z_0)f(z) = 0$. In fact, Conway takes this as the definition of removable singularity when f has an isolated singular point at $z = z_0$ (the converse of Theorem 6.77.2 is given below as Riemann's Theorem).

Note. To address the behavior of f near an essential singularity, we need the following lemma.

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Note. We mentioned Picard's Theorem at the end of Section 72, "The Three Types of Isolated Singular Points," which states that in any deleted neighborhood of an essential singularity, a function takes on every complex number an infinite number of times, with one exception. The following result is related to this idea, but falls short of Picard's Theorem. It states that in a deleted neighborhood of an essential singularity, a function gets arbitrarily close to every complex number.

Theorem 6.77.3. Casorati-Weierstrass Theorem.
Suppose that z_0 is an essential singularity of function f and let w_0 be any complex number. Then for all $\varepsilon > 0$, the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some point a in every deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 for $\delta > 0$.