Chapter 9. Conformal Mapping

Note. In this chapter, we consider mappings that preserve angles. We define harmonic functions and consider the interaction of conformal mappings and harmonic functions (and "boundary conditions"). These results will be applied in Chapter 10, "Applications of Conformal Mappings," to models related to temperature, electrostatic potential, and fluid flow.

Section 112. Preservation of Angles

Note. Consider a smooth arc $C = z(t)$ where $t \in [a, b]$ in the z-plane. Let $f(z)$ be analytic in an open neighborhood containing C. Then $f(z(t))$ is an arc in the w (image) plane. Call this arc Γ. For a particular $t_0 \in [a, b]$, suppose $z_0 = z(t_0)$ and $w_0 = f(z(t_0))$.

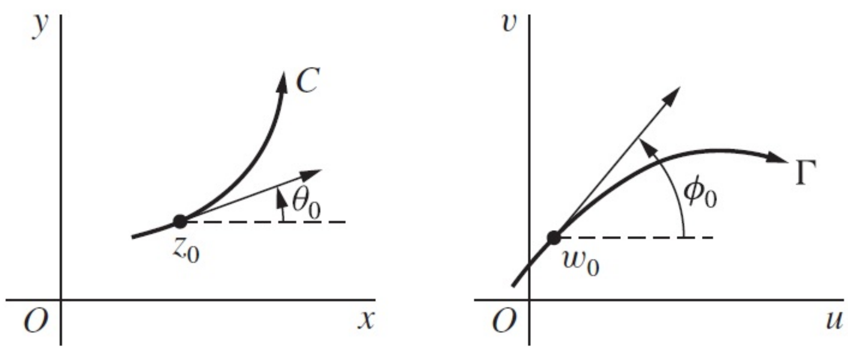


FIGURE 147
$\phi_0 = \psi_0 + \theta_0$.

Suppose $f'(z_0) \neq 0$. Since C is smooth, $z'(t_0)$ exists. Also,

$$w'(t_0) = f'(z(t_0))z'(t_0) = f'(z_0)z'(t_0).$$

Since $f'(z_0) \neq 0$ and $z'(t_0) \neq 0$, it follows that $w'(t_0) \neq 0$ and so Γ is smooth. Also,

$$\arg(w'(t_0)) = \arg(f'(z_0)) + \arg(z'(t_0)).$$

So the angle between a tangent to C at z_0 and a tangent to Γ at w_0 is $\arg(f'(z_0))$. Notice that this angle is independent of $C$and so if we take two smooth curves passing through z_0, say C_1 and C_2, with an angle of α between the directed tangent (directed since parameterized arcs are directed) of C_1 and the directed tangent C_2, then the angle between the images of the tangents to C_1 and the image of the tangent to C_2 will also be α.

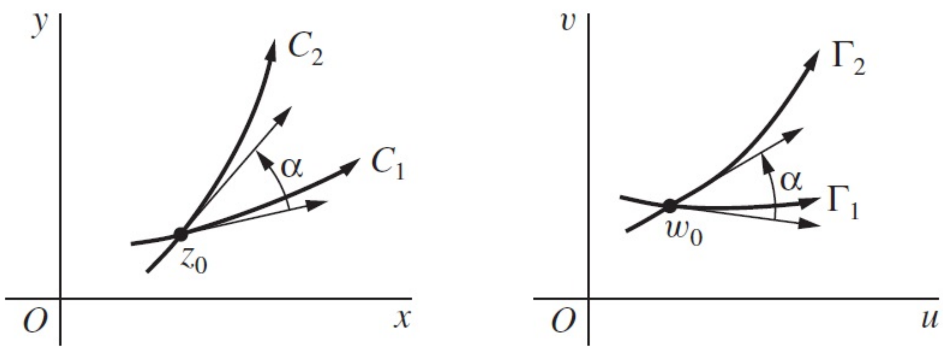


FIGURE 148

Definition. $f(z)$ is *conformal at* z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$. $f(z)$ is *conformal on a domain* D (open connected set) if it is conformal at each point of D.

Note. The above argument implies that a conformal map preserves both the magnitude and "sense" of the angle α.

Example 112.1. Consider $f(z) = e^z$. By definition, $f(z)$ is clearly conformal. Lets show it preserves angles in a special case. Let $C_1 : \text{Re}(z) = c_1$ and $C_2 : \text{Im}(z) = c_2$. See Figure 124 below left.

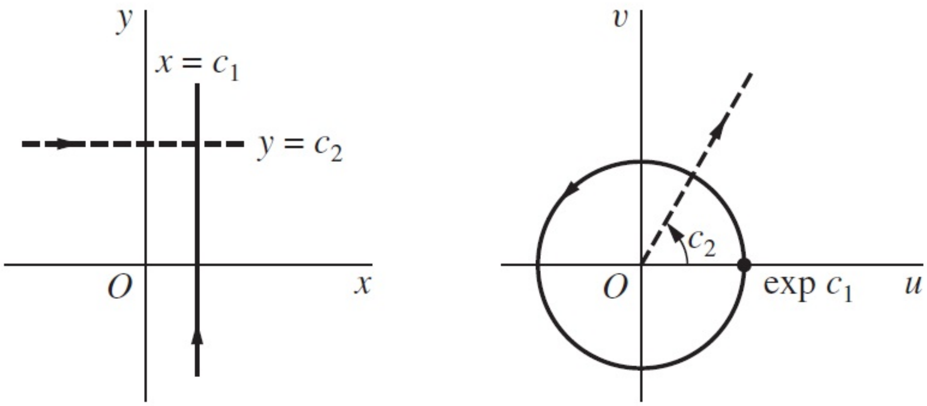


FIGURE 124
$w = \exp z$.

Then the angle between c_1 and c_2 is $\pi/2$ (we can treat c_1 and c_2 as having the given orientations). We can parameterize c_1 and c_2 as

$$C_1 : z_1(t) = c_1 + it \text{ and } C_2 : z_2(t) = t + ic_2.$$

Under $f(z) = e^z$:

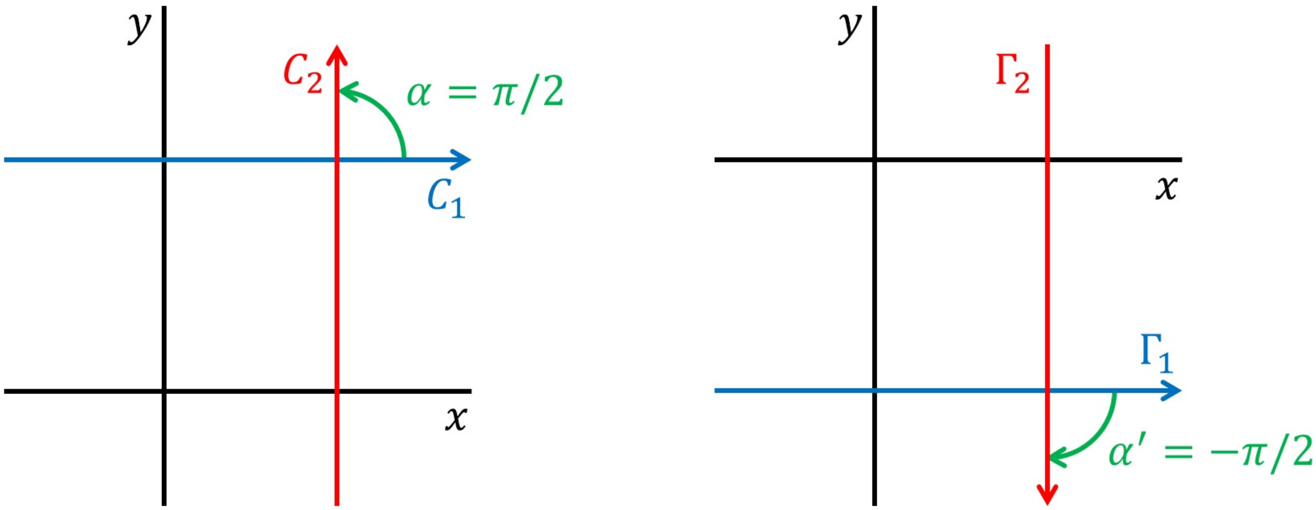
$$\Gamma_1 = \exp(C_1) : f(z_1(t)) = e^{z_1(t)} = e^{c_1+it} = e^{c_1}e^{it} \text{ and}$$

$$\Gamma_2 = \exp(C_2) : f(z_2(t)) = e^{z_2(t)} = e^{t+ic_2} = e^t e^{ic_2}.$$

These are given in Figure 124 right above. Notice that the angle of intersection of C_1 and C_2 is the same as the angle of intersection of $\Gamma_1 = \exp(C_1)$ and $\Gamma_2 = \exp(C_2)$ (and in the same sense).

Definition. A mapping that preserves the magnitude but not the sense of angles is an *isogonal mapping*.

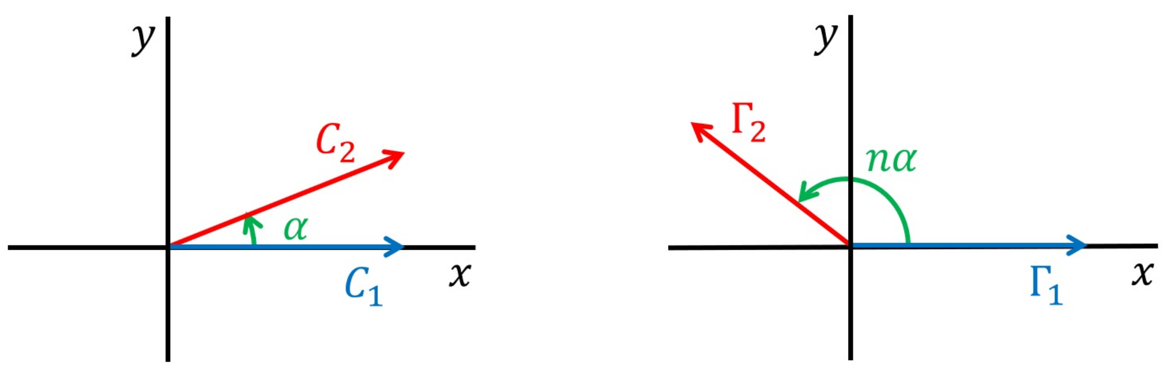
Example 112.3. $f(z) = \overline{z}$ is an isogonal mapping:



In fact, if f is conformal then $f(\overline{z})$ is isogonal.

Definition. Suppose that f is not a constant function and is analytic at a point z_0. If, in addition, $f'(z_0) = 0$, then z_0 is called a *critical point* of the transformation $w = f(z)$.

Note. $f(z) = z^n$ is not conformal at the critical point 0 since (for one thing):



Note. The ratio $\dfrac{|f(z)-f(z_0)|}{|z-z_0|}$ is the ratio of the distance from $f(z)$ to $f(z_0)$ and the distance from z to z_0. So this ratio gives an idea of how f scales distances. Of course,

$$|f'(z_0)| = \lim_{z\to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right|.$$

Definition. The *scale factor* of an analytic function at a point z_0 is $|f'(z_0)|$. If $|f'(z)| < 1$ on a domain D then f is a *contraction* on D and if $|f'(z)| > 1$ on D then f is an *expansion* on D.

Note. Contractions have a number of applications. See my online notes on Fundamentals of Functional Analysis (MATH 5740) on Section 2.12. Fixed Points and Contraction Mappings; notice the Contraction Mapping Theorem (Theorem 2.44).

Exercise 114.4. Show that the angle of rotation at a nonzero point $z_0 = r_0\exp(i\theta_0)$ under the transformation $f(z) = z^n$ ($n \in \mathbb{N}$). is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point.

Solution. With $z_0 = r_0\exp(i\theta_0)$, we have $f(z_0) = r_0^n\exp(in\theta_0)$ and so the angle of rotation is

$$\arg(f(z_0)) - \arg(z_0) = n\theta_0 - \theta_0 = (n-1)\theta_0.$$

For the scale factor, we have $f'(z) = nz^{n-1}$ so

$$|f'(z_0)| = |nz_0^{n-1}| = |n(r_0\exp(i\theta_0))^{n-1}| = nr_0^{n-1}|\exp(i(n-1)\theta_0)| = nr_0^{n-1}. \ \square$$

Revised: 1/18/2024