Section 113. Further Examples

Note. In this brief section we give two examples that illustrate how the preservation of angles and scales factors behave in different place in the complex plane.

Example 113.1. Consider $f(z) = z^2$ (or, equivalently, $f(x+iy) = x^2 - y^2 = 2xyi$). The derivative is $f'(z) = 2z$, so that $f'(z) \neq 0$ for $z \neq 0$. So be the definition of "conformal" in Section 112. Preservation of Angles and Scale Factors, f is conformal except at $z = 0$. For example, the half lines

$$C_1 : y = x \text{ for } x \geq 0, \text{ and } C_2 : x = 1 \text{ for } y \geq 0$$

intersect at $z_0 = 1 + i$. The angle from C_1 to C_2 is $\pi/4$ at the point of intersection z_0. See Figure 149 (left) below.

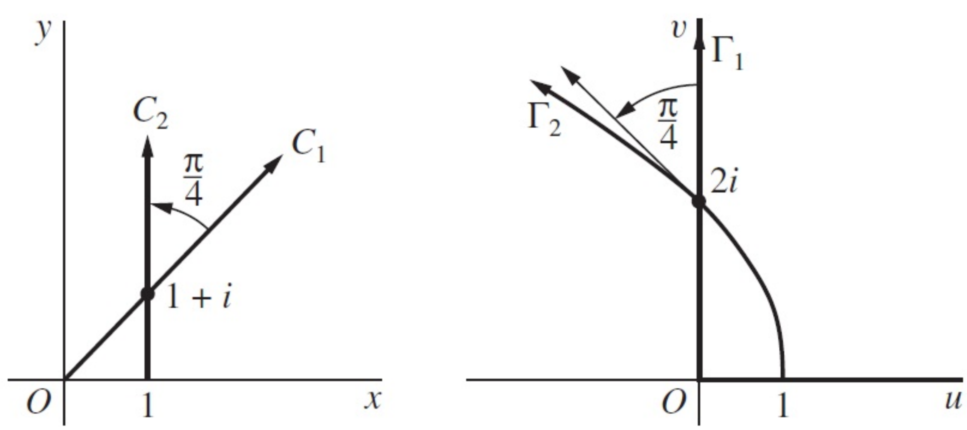

FIGURE 149
$w = z^2$.

Notice that $f(z_0) = f(1+i) = (1+i)^2 = 1 + 2i + i^1 = 2i$, so this is the point of intersection of the images of C_1 and C_2. In the $w = (u, v)$-plane, we have $w = f(z)$ has real part $u = x - y^2$ and imaginary part $v - 2xy$. The half line $C_1 : y = x$ for $x \geq 0$ is mapped to $u = x^2 - (x)^2 = 0$ $(x \geq 0)$ and $v = 2xy = 2x(x) = 2x^2$

$(x \geq 0)$. This is the upper half of the v axis as given in Figure 149 (right). We denote this image as Γ_1. The half line $C_2 : x = 1$ for $y \geq 0$ is mapped to $u = (1)^2 - y^2 = 1 - y^2$ $(y \geq 0)$ and $v = 2(1)y = 2y$ $(y \geq 0)$. This is the half parabola $v^2 = 4y^2 = 4(1-u)$ where $v \geq 0$, as given in Figure 149 (right). We denote this image as Γ_2. For Γ_2 we have u as a function of y $(u = 1 - y^2)$ and v as a function of y $(v = 2y)$. From the Chain Rule, $\dfrac{dv}{du}\dfrac{du}{dy} = \dfrac{dv}{dy}$, or $\dfrac{dv}{du} = \dfrac{dv/dy}{du/dy}$. Here we have $du/dy = -2y$ and $dv/dy = 2$, so $dv/du = (2)/(-2y) = -1/y$. In terms of x and y, the point of intersection is $x = 1$ and $y = 1$, so that at the point of intersection $dv/du = -1/(1) = -1$. We can also translate the derivative into terms of u and v, in which case we get $dv/du = -2/v$ which, at the point of intersection in terms of u and v (namely, $(u, v) = (0, 2)$), is also $dv/du = -2/(2) = -1$. In any case, the slope of the tangent to Γ_2 at $(0, 2)$ is -1. Therefore, the angle from Γ_1 to Γ_2 is $\pi/4$, computationally establishing the conformality of f at $z_0 = 1 + i$. Also notice that the scale factor at $z_0 = 1 + i$ is $|f'(1 + i)| = |2(1 + i)| = 2\sqrt{2}$.

Example 113.2. Consider the same half line C_2 from the previous example, and the C_3 the right-hand side of the real axis. See Figure 150 (left) below. The point of intersection this time is $z_0 = 1$ and we see that the angle from C_3 to C_2 is $\pi/2$. Again with $w = f(z) = z^2$, the image of C_2 is as it was in the previous example and the image of the right-hand side of the real axis under f is the positive real axis itself (f fixes the right-hand side of the real axis as a *set*, but only fixes the two *points* 0 and 1). Again, we see at the point of intersection of Γ_2 and Γ_3 (namely, 1) that the angle from Γ_3 to Γ_2 is $\pi/2$, illustrating the conformality of $f(z) = z^2$ at this second point. Notice that the scale factor at the point of intersection of C_2

and C_3 is $|f'(1)| = |2(1)| = 2$. This shows that the scale factor is different at $z_0 = 1$ from its value at $z_0 = 1 + i$ (from the previous example).

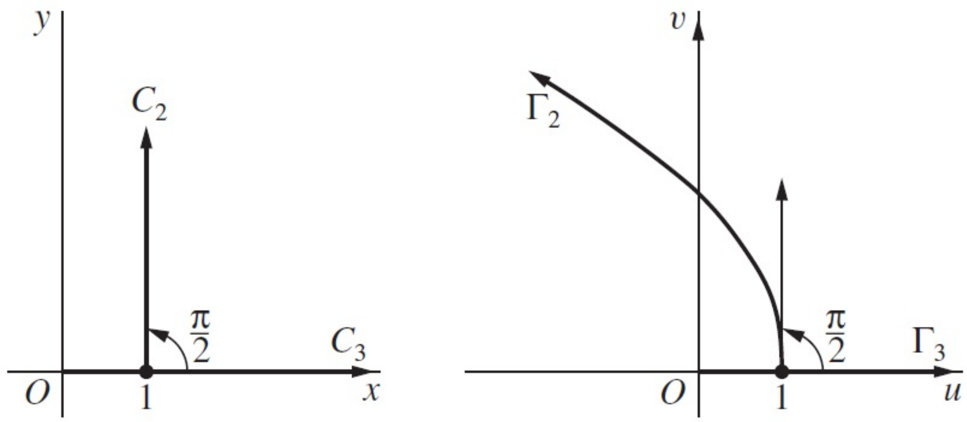

FIGURE 150
$w = z^2$.

Revised: 1/19/2024