Section 114. Local Inverses

Note. In this section, we use the Inverse Function Theorem from advanced calculus (or "Vector Calculus") to find inverse functions ("locally") of functions that satisfy certain differentiation properties. The differentiation property is satisfied by conformal mappings.

Note 114.A. Suppose $f(z)$ is analytic at z_0. Then f is analytic in some neighborhood of z_0. Let $z_0 = z_0 + iy_0$ and $f(z) = u(x,y) + iv(x,y)$. Then there is a neighborhood of (x_0, y_0) throughout which $u(x,y)$ and $v(x,y)$ and their partial derivatives of all orders are continuous (by, say, Theorem 5.57.A "Taylors Theorem"). If we let $u = u(x,y)$ and $v = v(x,y)$ then we can interpret these functions as mapping a region in the xy-plane into the uv-plane. Then the *Jacobian* of f is

$$\begin{aligned} J[f] &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y \\ &= u_x(u_x) - v_x(-v_x) \text{ by the Cauchy-Riemann Equations} \\ &= (u_x)^2 + (v_x)^2 = |f'(z)|^2. \end{aligned}$$

Note. To establish the main claim of this section, we need the Inverse Function Theorem. A general version is given in my online notes for Vector Calculus (formerly, "Vector Analysis" [MATH 4317/5317]) on Section 3.5. The Implicit Function Theorem. We do not need such a general case (which involves functions of n variables), so we state a special case involving functions of two variables.

Theorem 114.A. Inverse Function Theorem.

Let $U \subset \mathbb{R}^2$ be an open set and let $u : U \to \mathbb{R}$, and let $v : U \to \mathbb{R}$ have continuous partial derivatives. Let $f(x, y) = (u(x, y), v(x, y))$ and let $\mathbf{x}_0 = (x_0, y_0) \in U$. If $J[f(\mathbf{x}_0)] \neq 0$ then $f(\mathbf{x}_0) = \mathbf{s}_0$ can be solved uniquely as $\mathbf{x} = \mathbf{g}(\mathbf{s})$ for $\mathbf{x}$ near $\mathbf{x}_0$ (i.e., for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$) and $\mathbf{s}$ near $\mathbf{s}_0$ (i.e., for all $\mathbf{s}$ in some neighborhood of $\mathbf{s}_0$). Moreover, $\mathbf{g}$ has continuous partial derivatives.

Note 114.B. If f is a conformal mapping at $z_0 = x_0 + iy_0$ (that is, $f'(z_0) \neq 0$), then the Jacobian of f at z_0 is nonzero by Note 114.A: $J[f(z_0)] \neq 0$. Also as observed in Note 114.A, with $f = u + iv$, u and v have continuous partial derivatives (of all orders, in fact). So by the Inverse Function Theorem (Theorem 114.A), there are a pair of unique functions (the two components of function $\mathbf{g}$ in Theorem 114.A) $x = x(u, v)$ and $y = y(u, v)$ defined on some neighborhood N of (u_0, v_0) which have continuous partial derivatives, $u_0 = u(x_0, y_0)$ and $v_0 = v(x_0, y_0)$, and where

$$u = u(x, y) \text{ and } v = v(x, y) \text{ implies } x = x(u, v) \text{ and } y = y(u, v).$$

With $z = x + iy$, $w = u + iv$, $f(z) = u + iv$, and $g(w) = x + iy$, these two equations give

$$z = g(w) \text{ implies } w = f(z).$$

So we have $w = f(z) = f(g(w))$ for $w \in N$. This is to be used in Exercise 114.8 to show that $g'(w) = 1/f'(z)$ (once we know that g is analytic). With $w = u + iv$ and $w_0 = u_0 + iv_0$, and $g(w) = x(u, v) + iy(u, v)$, we consider the difference quotient for $(g(w_0 + \Delta w) - g(w_0))/\Delta w$. As shown in the proof of Theorem 2.21.A ("Differentiable

Implies the Cauchy-Riemann Equations"),

$$\lim_{\Delta w\to 0} \mathrm{Re}\left(\frac{g(w_0+\Delta w)-g(w_0)}{\Delta w}\right) = \lim_{\Delta u\to 0}\frac{x(u_0+\Delta u, v_0)-x(u_0,v_0)}{\Delta x} = x_u(u_0,v_0)$$

and

$$\lim_{\Delta w\to 0} \mathrm{Im}\left(\frac{g(w_0+\Delta w)-g(w_0)}{\Delta w}\right) = \lim_{\Delta u\to 0}\frac{y(u_0+\Delta u, v_0)-y(u_0,v_0)}{\Delta x} = y_u(u_0,v_0),$$

provided these limits exist, which they do as we know from the Inverse Function Theorem. These limits also hold for any point (u,v) in N, not just (u_0,v_0). Similarly,

$$\lim_{\Delta w\to 0} \mathrm{Im}\left(\frac{g(w_0+\Delta w)-g(w_0)}{\Delta w}\right) = y_v(u_0,v_0)$$

and

$$\lim_{\Delta w\to 0} \mathrm{Im}\left(\frac{g(w_0+\Delta w)-g(w_0)}{\Delta w}\right) = -x_v(u_0,v_0).$$

So computing these limits gives us x_u+iy_u and y_v-ix_v; these will be shown to be equal to each other and equal to $g'(w)$ in Exercise 114.7 where it is to be shown that g is analytic (using the equations in (5) below). Though we do not have $g'(w)=1/f'(z)$ for all $w\in N$, we can show that this relationship holds by taking limits similar to above. We know that f *is* analytic, so we have $f'(z)=u_x+iv_x=v_y-iu+y$. We can now conclude that

$$x_u+iy_u=\frac{1}{u_x+iv_x}=\frac{u_x-iv_x}{u_x^2+v_x^2}=\frac{1}{J}u_x+\frac{1}{J}(-iv_x)$$

and

$$y_v-ix_v=\frac{1}{v_y-iu_y}=\frac{v_y+iu_y}{v_y^2+u_y^2}=\frac{1}{J}v_y+\frac{1}{J}(iu_y)$$

(recall that the Cauchy-Riemann equations give, for analytic function f, $u_x=v_y$ and $v_x=-u_y$ so that $v_y^2+u_y^2=u_x^2+v_x^2$). We now have $x_u=(1/J)u_x=(1/J)v_y$,

$y_u = (-1/J)v_x$, $y_v = (1/J)v_y = (1/J)u_x$, and $x_v = (-1/J)u_y$. That is, we have throughout N:

$$x_u = \frac{1}{J}v_y, \qquad x_v = -\frac{1}{J}u_y, \qquad y_u = -\frac{1}{J}v_x, \qquad y_v = \frac{1}{J}u_x. \tag{5}$$

Note. The function $z = g(w)$ is the *local inverse* of $w = f(z)$ on neighborhood N. In conclusion, we have:

Theorem 114.B. If $w = f(z)$ is analytic and conformal at z_0 then there is an analytic function g defined on a neighborhood N of $f(z_0)$ such that $f(g(w)) = w$ for all w in N.

Example 114.1. Consider $f(z) = e^z$. Since $f'(z) = e^z$, then f is conformal in all of $\mathbb{C}$. Consider $z_0 = 2\pi i$. We have $w_0 = e^{z_0} = e^{2\pi i} = 1$. The local inverse of f at z_0 is $g(w) = \log w = \ln \rho + i\varphi$ where $w = \rho \exp(i\varphi)$ and we N as all such w with $\rho > 0$ and $\pi < \varphi < 3\pi$ (though other choices of N are possible; we cannot include 0 in N or take N as an annulus which "goes around" 0, or anything similar). A different choice of z_0 will potentially require a different choice of the values of φ in the local inverse. Notice that $g(1) = g(e^{2\pi i}) = \ln(1) + i(2\pi) = 2\pi i = z_0$. Also for $w = \rho \exp(i\varphi)$ where $\pi < \varphi < 3\pi$ (i.e., $w \in N$) we have

$$f(g(w)) = \exp(w) = \exp(\log z) = \exp(\ln \rho + i\varphi) = \exp(\ln \rho)\exp(i\varphi) = \rho \exp(i\varphi) = w.$$

Also,

$$g'(w) = \frac{d}{dw}[\log w] = \frac{1}{w} = \frac{1}{\exp z} = \frac{1}{f(z)},$$

as discussed above and to be shown in Exercise 114.8.