Section 115. Harmonic Conjugates

Note. In this section, we carry on with harmonic functions and complete the conversation start in Section 2.26. Harmonic Functions (from the 8th edition of the textbook).

Note. The next result appears in my online notes for Complex Variables (MATH 4337/5337) based on the 8th edition of Brown and Churchill's *Complex Variables and Applications*. See the notes on Section 2.26. Harmonic Functions and notice Theorem 2.26.1. That section also has examples illustrating the result.

Theorem 115.A. A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is harmonic conjugate of u.

Note. In the next theorem, we show that every harmonic function (on a simply connected domain) always has a harmonic conjugate on the domain. The result is also addressed in graduate-level Complex Analysis 1 (MATH 5510) in Section III.2. Analytic Functions; see Theorem III.2.30.

Theorem 115.B. If a harmonic function $u(x, y)$ is defined on a simply connected domain D, it always has a harmonic conjugate $v(x, y)$ in D.

Note 115.A. We have from the proof of Theorem 115.B that the harmonic conjugate of harmonic function u is

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} (-u_t(s,t)\,ds + u_s(s,t)\,dt).$$

We have proved that v is the harmonic conjugate of u, but that does not mean that we can evaluate the integral to get $v(x,y)$ in a "nice" form. Remember, integration is hard!

Example 115.3. Consider the function $u(x,y) = 2x - 2xy$. We have $u_x = 2 - 2y$ so that $u_{xx} = 0$, and $u_y = -2x$ so that $u_{yy} = 0$. Therefore u is harmonic in all of $\mathbb{C}$. By Theorem 115.B, we know that u has a harmonic conjugate v. We have $-u_t(s,t) = -(-2x)|_{(x,y)=(s,t)} = 2s$ and $u_s(s,t) = (2-2y)|_{(x,y)=(s,t)} = 2 - 2t$. We choose $(x_0,y_0) = (0,0)$ and we now have from Note 115.A that

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} (-u_t(s,t)\,ds + u_s(s,t)\,dt) = \int_{(0,0)}^{(x,y)} (2s\,ds + (2-2t)\,dt)$$

$$= \int_{(0,0)}^{(x,y)} 2s\,ds + \int_{(0,0)}^{(x,y)} (2-2t)\,dt = s^2|_{s=x} + (2t - t^2)|_{t=y} = x^2 + (2y - y^2).$$

We can add an any constant to get the general form of a harmonic conjugate of u: $v(x,y) = x^2 + 2y - y^2 + C$.

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