Section 116. Transformations of Harmonic Functions

Note. In this section, we consider harmonic functions which, later, will play a role in solving certain partial differential equations (or "PDEs"). An ordinary differential equation comes either with an initial value at one point, or boundary values at two points, since the unknown function is a function of a single variable, $y = f(x)$. See my online notes for Differential Equations (MATH 2120) on Section 1.3. Initial-Value Problems, Boundary-Value Problems, and Existence of Solutions. If we consider a function of two variables, then rates of change will be represented by partial derivatives. The PDE will hold on some region in the plane, and the boundary conditions will involve the edge of the region (it will be some continuous curve bounding the region). We will consider temperature distributions, electrostatic potential, and fluid flow in two-dimensions in Chapter 10, "Applications of Conformal Mapping."

Note/Definition. A PDE for which the value of the unknown function is prescribed on the boundary of the region, the resulting boundary value problem (or"BVP") is a *boundary value problem of the first kind*, or a *Dirichlet problem*. The boundary condition is called a *Dirichlet condition*; see my online notes for Applied Mathematics 2 (MATH 5620) on Section 4.1. Separation of Variables, The Dirichlet Condition. A PDE for which the normal derivative of the unknown function is prescribed on the boundary of the region is a *boundary value problem of the second kind*, or a *Neumann problem*. The normal derivative would reflect the flow through the boundary. For example, if $u(x, y)$ describes the temperature of a flat plate at different points (x, y) (think of the temperature as a measurement of the "amount" of heat), and some part of the boundary is insulated, then the normal derivative on that part of the boundary would be zero. Such BVPs are explored in the Applied Mathematics 2 class in Section 4.2. The Neumann Condition. We will consider PDEs which have harmonic functions as solutions, and in this way we will use real and imaginary parts of analytic functions and their behaviors on boundaries. The following theorem will prove useful in such applications. It shows that a harmonic function transforms the real and imaginary parts of an analytic function to a harmonic function (thus the title of this section). This will allow us to transfer certain "awkward" regions to "nicer" regions. Finding harmonic functions on the nice region will be used to yield harmonic functions on the awkward regions. To solve BVPs, we'll need to also consider how boundary conditions transform (this is the topic of the next section).

Theorem 116.A. Suppose that

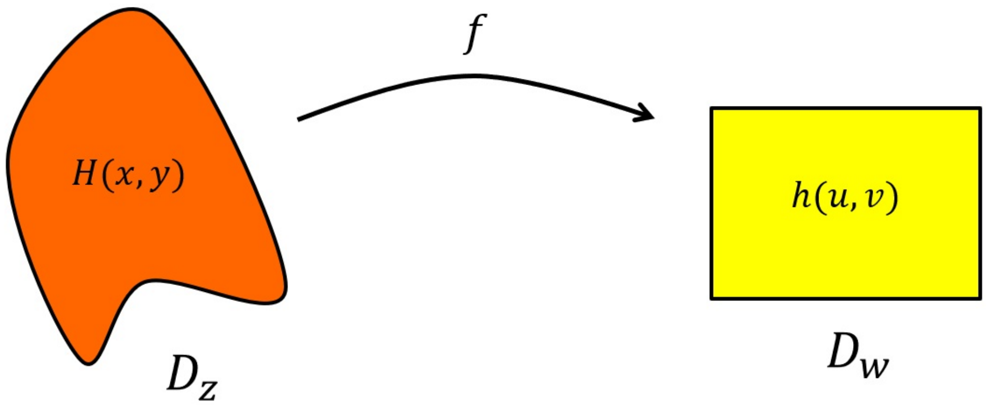
(a) an analytic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_z in the z plane onto a domain D_w in the w plane;

(b) $h(u, v)$ is a harmonic function defined on D_w.

It follows that the function $H(x, y) = h[(u(x, y), v(x, y)]$ is harmonic in D_z.

Note. The second part of the proof of Theorem 116.A, where D_w is not simply connected, can be shown directly by computing second partial derivatives with the Chain Rule, as is to be shown in Exercise 117.8.

Note 116.A. We will use Theorem 116.A to transform harmonic functions from one setting to another. We will relate a given PDE in one "difficult" region to a PDE in another "easier" region. We find a solution over the easier region and then use it to "pull back" to a solution in the difficult region:



Example 116.3. To illustrate the use of Theorem 116.A, consider the analytic function $w = f(z) = \operatorname{Log} z = \ln r + i\Theta$ where $r > 0$ and $-\pi/2 < \Theta < \pi/2$. In rectangular coordinates it is of the form $w = \operatorname{Log} z = \ln \sqrt{x^2 + y^2} + i \arctan(y/x)$, where $-\pi/2 < \arctan t < \pi/2$ (as is standard). This transformation maps the right half plane (D_z in the notation of the theorem) onto the horizontal strip $-\pi/2 < v < \pi/2$ (D_w in the notation of the theorem), as is to be shown in Exercise 117.3 with the help of the following figure:

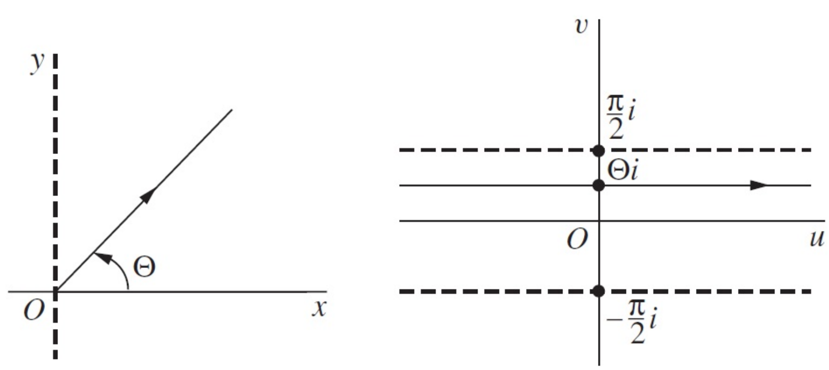


FIGURE 153
$w = \operatorname{Log} z$.

Now the function $h(u,v) = \operatorname{Im} w = v$ is harmonic if that strip (since both second partials are 0), so Theorem 116.A implies that the function

$$H(x,y) = h(u(x,y),v(x,y)) = h(\ln\sqrt{x^2+y^2}, \arctan(y/x)) = \arctan(y/x)$$

is harmonic in the half plane $x > 0$.

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