Section 117. Transformations of Boundary Conditions

Note. In this section, we consider transformations of boundary conditions associated with Dirichlet problems and Neumann problems. In the next chapter this, combined with the results of Section 116. Transformations of Harmonic Functions, will allow us to transform a given BVP in the xy plane to a (simpler) one in the uv plane. We'll then solve the simpler BVP and use it to "pull back" to a solution of the original BVP.

Definition/Note 117.A. We need to review gradients and directional derivatives, topics covered in Calculus 3 (MATH 2110); see my online notes for that class on Section 14.5. Directional Derivatives and Gradient Vectors. For function $f(x, y)$, the *gradient vector* is the two dimensional vector $\nabla f = (\partial f/\partial x)\mathbf{i}+(\partial f/\partial y)\mathbf{j}$. When evaluated at a point P_0, the gradient points in the direction of greatest increase of the function $f(x, y)$. The magnitude of the gradient at P_0 reflects the steepness of a tangent to the surface $z = f(z, y)$ in the direction of the gradient. The *directional derivative* of $f(x, y)$ at point P_0 in the direction of two dimensional unit vector $\mathbf{u}$ is $(df/ds)|_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}$. The value of this directional derivative gives the slope of a tangent line to the surface $z = f(x, y)$ in the direction $\mathbf{u}$.

Theorem 117.A. Suppose that

(a) a transformation $w = f(z) = u(x, y) + iv(x, y)$ is conformal at each point of a smooth arc C and that Γ is the image of C under that transformation;

(b) $h(u, v)$ is a function that satisfies one of the conditions $h = h_0$ and $dh/dn = 0$ at points on Γ, where h_0 is a real constant and dh/dn denotes the directional derivatives of h normal to Γ.

It follows that the function $H(x, y) = h[(u(x, y), v(x, y)]$ satisfies the corresponding condition $H = h_0$ or $dH/dN = 0$ at points on C, where dH/dN denotes directional derivatives of H normal to C.

Note 117.A. In the proof of Theorem 117.A, we assumed $\nabla h \neq \mathbf{0}$. We now address this case (thus adding the missing detail in the proof). In Exercise 9.117.10(a), it is shown that $\|\nabla H(x, y)\| = \|\nabla h(u, v)\||f'(z)|$. So for $\nabla h = \mathbf{0}$, we have $\nabla H = \mathbf{0}$. So the directional derivatives dh/dn and dH/dN satisfy $dh/dn = (\nabla h) \cdot \mathbf{n} = 0$ and $dH/dN = (\nabla H) \cdot \mathbf{N} = 0$, as claimed in the conclusion of Theorem 117.A.

Note 117.B. Brown and Churchill have omitted some hypotheses from Theorem 117.A. They have assumed that (a) ∇h and ∇H always exist (since the directional derivatives require this), and (b) the level curve $H(x, y) = c$ is smooth when $\nabla h \neq \mathbf{0}$ at (u, v). The smoothness of curve $H(x, y) = c$ implies that the partial derivatives of H exist and are continuous (remember that "smooth" means continuously differentiable; see my online notes for Calculus 2 [MATH 1920] on Section 6.3. Arc Length). So that for $\nabla h \neq \mathbf{0}$ on C we have that ∇h exists and is nonzero so that angles between curves (C and c, and Γ and c) are defined and preserved by conformal mapping $w = f(z)$. These extra conditions will be met in all the applications we consider in Chapter 10.

Example 117.A. To illustrate Theorem 117.A, consider $h(u,v) = v+2$. Let transformation $w = f(z) = f(x+iy)$ be

$$w = iz^2 = i(z+iy)^2 = -2xy + i(x^2 - y^2)$$

is conformal when $z \neq 0$ (since $f'(z) = 2iz$). Consider the half line $y = x$ with $x > 0$. Notice the principal argument of each complex number on the half line is $\pi/4$. So $f(z) = iz^2$ maps this half line to the negative u-axis (where the principal argument of each complex number is π and this is coterminal with $\pi/2 + 2(\pi/4)$). Since $v = 0$ on the u-axis, then $h(u,v) = v+2$ takes on the value $h = 2$ on the negative u-axis. In addition, the positive x-axis (where $y = 0$) is mapped by $w = f(z)$ to the positive v-axis (since $w = ix^2$ where $x > 0$). On the positive v-axis (where $u = 0$) we have the partial derivative $h_u = 0$; that is, the normal derivative of h on the positive v-axis is 0 (see Figure 152, right).

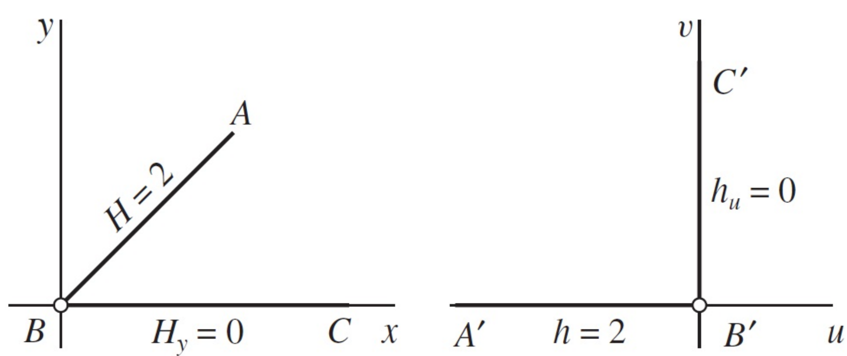

FIGURE 152

Since $w = u + iv = -2xy + i(x^2 - y^2)$, then $u(x,y) = -2xy$ and $v(x,y) = x^2 - y^2$. With $h(u,v) = v + 2$, we have

$$H(x,y) = H(u(x,y), v(x,y)) = H(-2xy, x^2 - y^2) = (x^2 - y^2) + 2 = x^2 - y^2 + 2.$$

By Theorem 117.A, we must have $H = 2$ on the half line $y = x$ where $x > 0$ (as is confirmed by substitution) since this is mapped by f to the negative u-axis where

$h = 2$. Also by Theorem 117.A, the normal derivative H_y must be 0 on the positive x-axis (as is confirmed by differentiation and substitution) since this is mapped by f to the positive v-axis and the normal derivative $h_u = 0$ on the positive v-axis. See Figure 152 again for these conditions.

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