Supplement. Graph Decompositions, Packings, and Coverings

Note. In this supplement, we consider a simple graph $G$ (often $K_n$) and various isomorphic decompositions into copies of a given small graph. We start with Steiner triple systems.

Definition. A Steiner triple system of order $n$ is an isomorphic decomposition of $G = K_n$ into a family $\mathcal{F}$ of subgraphs of $G$ such that each $F \in \mathcal{F}$ is isomorphic to a 3-cycle. The members of $\mathcal{F}$ are called the blocks of the Steiner triple system. We denote a Steiner triple system of order $n$ as a $STS(n)$.

Note. The definition of a Steiner triple system in the realm of design theory is slightly different (though equivalent) to the above. Quoting from Lindner and Rodger’s Design Theory [1](see page 1):

A Steiner triple system is an ordered pair $(S, T)$ where $S$ is a finite set of points or symbols, and $T$ is a set of 3-element subsets of $S$ called triples, such that each pair of distinct elements of $S$ occurs together in exactly one triple of $T$. The order of a Steiner triple system $(S, T)$ is the size of the set $S$, denoted $|S|$.

Note. Lindner and Rodger [1, page 1] say that Steiner triple systems were apparently first defined by W. S. B. Wool-House in 1844 in the Lady’s and Gentlemen’s
Diary as “Prize Question 1733.” The problem was ultimately solved by Thomas P. Kirkman (1806–1895) in “On a Problem of Combinations,” *Cambridge and Dublin Mathematics Journal*, 2 (1847), 191–204. Ironically, Steiner triple systems are named for Jakob Steiner (1796–1863), a Swiss mathematician working in Berlin most of his career, who gave necessary conditions for their existence and published it in “Combinatorische Aufgabe,” *Journal für die Reine und angewandte Mathematik (Crelles Journal)*, 45 (1853), 181–182. The strange dates on the necessary conditions of Steiner and the sufficiency of Kirkman are explained by a lack of communication between mainland Europe and the British Isles at the time—this likely results from fallout from the argument between Newton and Leibniz over credit for calculus.

These images are from the MacTutor History of Mathematics Archive (accessed 3/6/2020).

**Theorem.** (Kirkman, 1847) A Steiner triple system of order $n$ exists if and only if $v \equiv 0$ or 1 (mod 3).
Examples. Let the vertex set of $K_7$ be \{0, 1, 2, 3, 4, 5, 6\}. We represent the 3-cycle on vertices $a, b, c$ with edges $\{a, b\}, \{b, c\}, \text{and} \{a, c\}$ as $(a, b, c) = (b, c, a) = (c, a, b)$. Then the blocks of a Steiner triple system of order 7 is given by:

$$\mathcal{F} = \{(0, 1, 3), (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 0), (5, 6, 1), (6, 0, 2)\}.$$  

With the vertex set of $K_9$ as \{0, 1, 2, \ldots, 7, 8\}, the blocks of a Steiner triple system of order 9 is given by:

$$\mathcal{F} = \{(0, 1, 2), (0, 3, 6), (0, 4, 8), (0, 5, 7), (3, 4, 5), (1, 4, 7), (1, 5, 6), (1, 3, 8), (6, 7, 8), (2, 5, 8), (2, 3, 7), (2, 4, 6)\}.$$  

There is a clear pattern in the blocks of the $STS(7)$, but the pattern for the $STS(9)$ is not clear.

Note. Let’s establish the necessary conditions first given by Steiner. The argument is based on the number of edges and the degrees of vertices.

Lemma. If $STS(n)$ exists then $n \equiv 0$ or 1 (mod 6).

Proof. The graph $K_n$ has \( \binom{n}{2} = \frac{n(n-1)}{2} \) total edges and a 3-cycle has 3 edges. Now the edges sets of the 3-cycles in a Steiner triple system partition the edge set of $K_n$, so it is necessary that 3 divides $\frac{n(n-1)}{2}$; that is, we must have $n(n-1) \equiv 0$ (mod 6). So we must have $n \equiv 0$ or 1 (mod 3). Next, the degree of each vertex in $K_n$ is $n - 1$ and the degree of each vertex of a 3-cycle is 2, so we must also have $n - 1$ even, or $n$ odd. Therefore, it is necessary that $n \equiv 1$ or 3 (mod 6).
References


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