Lemma 13.1.

Let \( X \) be a set and let \( \mathcal{B} \) be a basis for a topology \( \mathcal{T} \) on \( X \).

Then \( \mathcal{T} \) equals the collection of all unions of elements of \( \mathcal{B} \). Hence \( \mathcal{T} \cap \mathcal{B} \), the topology \( \mathcal{B} \) generated by \( \mathcal{B} \), is also a topology on \( X \). Since \( \mathcal{B} \in \mathcal{T} \), then \( \mathcal{B} \) is a basis for \( \mathcal{B} \) generated by \( \mathcal{B} \).

Proof. By the definition of \( \mathcal{T} \), every \( U \in \mathcal{T} \) can be written as a union of elements of \( \mathcal{B} \). For each \( U \in \mathcal{T} \), choose \( B \in \mathcal{B} \) such that \( U = \bigcup B \).

Theorem 13.4. Let \( \mathcal{B} \) be a basis for a topology \( \mathcal{T} \) on \( X \).

Define \( \mathcal{T} = \{ \cap \mathcal{B} \mid \mathcal{B} \subseteq \mathcal{T} \} \) where \( \mathcal{B} \subseteq \mathcal{T} \) for some \( \mathcal{B} \in \mathcal{B} \). Then \( \mathcal{T} \) is a topology on \( X \).

Proof. We consider the definition of \( \mathcal{T} \).

The \( \mathcal{T} \) generated by \( \mathcal{B} \). Then \( \mathcal{T} \) is a topology on \( X \).

Since \( \mathcal{B} \subseteq \mathcal{T} \), then \( \mathcal{T} \) is a topology on \( X \).

Therefore, \( \mathcal{T} \) is a topology on \( X \).
Theorem 13.2 (contd).

Lemma 13.3

Definition of topology (B part (3)). By part (3) of the definition of topology, the collection of open sets of $x$ is a basis for topology on $X$, where $x \in X$.

Proof. First we show that $\mathcal{C}$ is a basis. For the first part of the definition of topology, let $x, y \in X$. Then there is a collection of open sets of $x$ that contains open set $\mathcal{C}$ and a collection of open sets of $y$ that contains open set $\mathcal{C}$. Therefore, $X$ is a topological space. Suppose that $x$ is a

Lemma 13.4

Therefore, $X$ is a topological space.

Proof. Let $x, y \in X$. Then there is a collection of open sets of $x$ that contains open set $\mathcal{C}$ and a collection of open sets of $y$ that contains open set $\mathcal{C}$. Therefore, $X$ is a topological space.
Let \( L \) be the set of all finite intersections of elements of \( S \). Then \( L \) is a topology on the set \( X \). Therefore, \( L \) is a basis for a topology on \( X \). By Lemma 13.1, the topology generated by \( L \) is the topology \( \tau \) on \( X \), consisting of all unions of elements of \( L \). This is precisely the collection of sets in \( L \). So \( L \) is a topology on \( X \). The topology \( \tau \) for which \( L \) is a basis is the topology defined as follows. For any \( S \subseteq \mathcal{S} \), where \( \mathcal{S} \) is a basis for the topology on \( X \), let \( \{ S \in \mathcal{S} \mid \bigcup S \subseteq \bigcup S' \} = L \).

**Proof.** Let \( L \) be the set of all finite intersections of elements of \( S \). Then \( L \) is a topology on \( X \). Let \( S \) be a basis for a topology on \( X \). Define \( L \) to be