Chapter 2. Topological Spaces and Continuous Functions
Section 14. The Order Topology—Proofs of Theorems
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**Theorem 14.A.** Let $X$ be a set with a simple order relation and let $\mathcal{B}$ consist of all open intervals $(a, b)$, all intervals $[a_0, b)$, and all intervals $(a, b_0]$, where $a_0$ is the least element of $X$ and $b_0$ is the greatest element of $X$ (if such exist). Then $\mathcal{B}$ is a basis for a topology on $X$.

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Theorem 14.B

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**Proof.** Let $S$ be the set of all open rays. As observed above, the open rays are in fact open sets in the order topology, so $S \subset \mathcal{T}$ and the topology generated by $S$ is a subset of $\mathcal{T}$ as well (Lemma 31.1).
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(Notice that if $a_0$ is the least and $b_0$ is the greatest element of $X$ then $[a_0, +\infty) = (-\infty, b)$ and $(a, b_0] = (a, +\infty)$ are in fact open rays.)
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