Chapter 2. Topological Spaces and Continuous Functions
Section 16. The Subspace Topology—Proofs of Theorems
Table of contents

1. Lemma 16.1
2. Lemma 16.2
3. Lemma 16.3
4. Theorem 16.4
Lemma 16.1. If $\mathcal{B}$ is a basis for the topology of $X$ then the set $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B} \}$ is a basis for the subspace topology on $Y$.

Proof. Let $U$ be open in $X$ so that $U \cap Y \in \mathcal{B}_Y$. Let $y \in U \cap Y$. Then since $\mathcal{B}$ is a basis for the topology of $X$, there is (open) $B \in \mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap y \subset U \cap Y$. Then by Lemma 13.2, $\mathcal{B}_Y$ is a basis for the subspace topology on $Y$. 
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Lemma 16.2. Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$, then $U$ is open in $X$.

**Proof.** Let $U$ be open in $Y$. Then $U = Y \cap V$ for some set $V$ open in $X$. Since $Y$ and $V$ are both open in $X$, then $Y \cap V = U$ is open in $X$. \qed
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Lemma 16.3

Lemma 16.3. If $A$ is a subspace of $X$ and $B$ is a subspace of $Y$, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. Let $U \times V$ be a basis element for the product topology on $X \times Y$. Then $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$. 


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Proof. Let $U \times V$ be a basis element for the product topology on $X \times Y$. Then $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$. Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$. Since $U \cap A$ and $V \cap B$ are open relative to $A$ and $B$, respectively, then $(U \cap A) \times (V \cap B)$ is a basis element for the product topology on $A \times B$. So the basis for the subspace topology on $A \times B$ is a subset of the basis for the product topology on $A \times B$. 
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Theorem 16.4

**Theorem 16.4.** Let $X$ be an ordered set in the order topology. Let $Y$ be a subset of $X$ that is convex in $X$. Then the order topology on $Y$ is the same as the subspace topology on $Y$.

**Proof.** By Theorem 14.B, the set of all open rays form a subbasis for the order topology on $X$. Then the set

$B_S = \{(a, +\infty) \cap Y, Y \cap (-\infty, a) \mid a \in X\}$

is a subbasis for the subspace topology on $Y$. Since $Y$ is convex then for $a \in Y$ we have

$(a, +\infty) \cap Y = \{x \in Y \mid x > a\}$

and

$(−\infty, a) \cap Y = \{x \in Y \mid x < a\}$

and each of these is an open ray in $Y$. If $a/\in Y$ then these two sets are either $\emptyset$ or $Y$. In all cases, each is open in the order topology and so the subspace topology is a subset of the subspace topology.

Conversely, any open ray of $Y$ equals the intersection of an open ray of $X$ with $Y$ and so is open in the subspace topology on $Y$. Since the open rays of $Y$ are a subbasis for the order topology on $Y$ by Theorem 14.B, this topology is a subset of the subspace topology. Therefore, the subspace topology on $Y$ is the same as the order topology on $Y$. 

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Introduction to Topology
May 31, 2016 6 / 6
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