Introduction to Topology

Chapter 2. Topological Spaces and Continuous Functions
Section 18. Continuous Functions—Proofs of Theorems
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Lemma 18.A

Lemma 18.A. Let $f : X \to Y$, let $\mathcal{B}$ be a basis for the topology on $Y$, and let $\mathcal{S}$ be a subbasis for the topology on $Y$.

1. $f$ is continuous if $f^{-1}(B)$ is open in $X$ for each $B \in \mathcal{B}$.
2. $f$ is continuous if $f^{-1}(S)$ is open in $X$ for each $X \in \mathcal{S}$.

Proof. (1) Let $V \subset Y$ be open. Then (by definition of basis) there are $B_\alpha \in \mathcal{B}$ for $\alpha \in J$ such that $V = \bigcup_{\alpha \in J} B_\alpha$. 


Lemma 18.A. Let \( f : X \to Y \), let \( \mathcal{B} \) be a basis for the topology on \( Y \), and let \( \mathcal{S} \) be a subbasis for the topology on \( Y \).

(1) \( f \) is continuous if \( f^{-1}(B) \) is open in \( X \) for each \( B \in \mathcal{B} \).

(2) \( f \) is continuous if \( f^{-1}(S) \) is open in \( X \) for each \( X \in \mathcal{S} \).

Proof. (1) Let \( V \subset Y \) be open. Then (by definition of basis) there are \( B_\alpha \in \mathcal{B} \) for \( \alpha \in J \) such that \( V = \bigcup_{\alpha \in J} B_\alpha \). Then \( f^{-1}(V) = f^{-1}(\bigcup_{\alpha \in J} B_\alpha) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha) \) is an open set in \( X \) by hypothesis. So each \( f^{-1}(B_\alpha) \) is open in \( X \) and \( f^{-1}(V) \) is open in \( X \). Hence \( f \) is continuous.
Lemma 18.A. Let $f : X \to Y$, let $\mathcal{B}$ be a basis for the topology on $Y$, and let $\mathcal{S}$ be a subbasis for the topology on $Y$.

1) $f$ is continuous if $f^{-1}(B)$ is open in $X$ for each $B \in \mathcal{B}$.

2) $f$ is continuous if $f^{-1}(S)$ is open in $X$ for each $X \in \mathcal{S}$.

Proof. (1) Let $V \subset Y$ be open. Then (by definition of basis) there are $B_\alpha \in \mathcal{B}$ for $\alpha \in J$ such that $V = \bigcup_{\alpha \in J} B_\alpha$. Then $f^{-1}(V) = f^{-1}(\bigcup_{\alpha \in J} B_\alpha) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$ is an open set in $X$ by hypothesis. So each $f^{-1}(B_\alpha)$ is open in $X$ and $f^{-1}(V)$ is open in $X$. Hence $f$ is continuous.
Lemma 18.A. Let $f : X \to Y$, let $B$ be a basis for the topology on $Y$, and let $S$ be a subbasis for the topology on $Y$.

(1) $f$ is continuous if $f^{-1}(B)$ is open in $X$ for each $B \in B$.

(2) $f$ is continuous if $f^{-1}(S)$ is open in $X$ for each $X \in S$.

Proof (continued). (2) Let $V \subset Y$ be open. Then (by the definition of subbasis) there are $S^i_\alpha$ for $\alpha \in J$, $i \in \mathbb{N}$ such that $V = \bigcup_{\alpha \in J} (S^1_\alpha \cap S^2_\alpha \cap \cdots \cap S^{n_\alpha}_\alpha)$. 


**Lemma 18.A**. Let \( f : X \to Y \), let \( \mathcal{B} \) be a basis for the topology on \( Y \), and let \( \mathcal{S} \) be a subbasis for the topology on \( Y \).

1. \( f \) is continuous if \( f^{-1}(B) \) is open in \( X \) for each \( B \in \mathcal{B} \).
2. \( f \) is continuous if \( f^{-1}(S) \) is open in \( X \) for each \( S \in \mathcal{S} \).

**Proof (continued).** (2) Let \( V \subset Y \) be open. Then (by the definition of subbasis) there are \( S^i_\alpha \) for \( \alpha \in J \), \( i \in \mathbb{N} \) such that \( V = \bigcup_{\alpha \in J} (S^1_\alpha \cap S^2_\alpha \cap \cdots \cap S^{n_\alpha}_\alpha) \). Then

\[
 f^{-1}(V) = f^{-1}(\bigcup_{\alpha \in J} (S^1_\alpha \cap S^2_\alpha \cap \cdots \cap S^{n_\alpha}_\alpha)) = \bigcup_{\alpha \in J} f^{-1}(S^1_\alpha \cap S^2_\alpha \cap \cdots \cap S^{n_\alpha}_\alpha)
\]

\[
 = \bigcup_{\alpha \in J} (f^{-1}(S^1_\alpha) \cap f^{-1}(S^2_\alpha) \cap \cdots \cap f^{-1}(S^{n_\alpha}_\alpha))
\]

is open in \( X \) since each \( f^{-1}(S^i_\alpha) \) is open in \( X \) by hypothesis and so \( f^{-1}(S^1_\alpha) \cap f^{-1}(S^2_\alpha) \cap \cdots \cap f^{-1}(S^{n_\alpha}_\alpha) \) is open for each \( \alpha \in J \), and hence the union is open. So \( f^{-1}(V) \) is open and \( f \) is continuous.
Lemma 18.A (continued)

**Lemma 18.A.** Let \( f : X \to Y \), let \( B \) be a basis for the topology on \( Y \), and let \( S \) be a subbasis for the topology on \( Y \).

1. \( f \) is continuous if \( f^{-1}(B) \) is open in \( X \) for each \( B \in B \).
2. \( f \) is continuous if \( f^{-1}(S) \) is open in \( X \) for each \( X \in S \).

**Proof (continued).** (2) Let \( V \subset Y \) be open. Then (by the definition of subbasis) there are \( S_i^\alpha \) for \( \alpha \in J, \ i \in \mathbb{N} \) such that

\[
V = \bigcup_{\alpha \in J} (S_\alpha^1 \cap S_\alpha^2 \cap \cdots \cap S_\alpha^n).
\]

Then

\[
f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha \in J} (S_\alpha^1 \cap S_\alpha^2 \cap \cdots \cap S_\alpha^n)\right) = \bigcup_{\alpha \in J} f^{-1}(S_\alpha^1 \cap S_\alpha^2 \cap \cdots \cap S_\alpha^n)
\]

\[
= \bigcup_{\alpha \in J} (f^{-1}(S_\alpha^1) \cap f^{-1}(S_\alpha^2) \cap \cdots \cap f^{-1}(S_\alpha^n))
\]

is open in \( X \) since each \( f^{-1}(S_\alpha^i) \) is open in \( X \) by hypothesis and so

\[
f^{-1}(S_\alpha^1) \cap f^{-1}(S_\alpha^2) \cap \cdots \cap f^{-1}(S_\alpha^n) \]

is open for each \( \alpha \in J \), and hence the union is open. So \( f^{-1}(V) \) is open and \( f \) is continuous. \( \square \)
Theorem 18.1. Let $X$ and $Y$ be topological spaces. let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $Z$ of $X$, one has $f(A) \subseteq f(A)$.
3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subseteq V$.

Proof. (1)$\Rightarrow$(2) Suppose $f$ is continuous. Let $A \subseteq X$ and $x \in \overline{A}$. If $x \in A$ then $f(x) \in f(A) \subseteq f(A)$. 


Theorem 18.1

Theorem 18.1. Let $X$ and $Y$ be topological spaces. let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $Z$ of $X$, one has $f(\overline{A}) \subset \overline{f(A)}$.
3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

Proof. (1)$\Rightarrow$(2) Suppose $f$ is continuous. Let $A \subset X$ and $x \in \overline{A}$. If $x \in A$ then $f(x) \in f(A) \subset \overline{f(A)}$. If $x \notin A$ then let $V$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open in $X$ and $x \in f^{-1}(V)$. By definition of $\overline{A}$, $f^{-1}(V)$ intersects $A$ at some point $y \neq x$. 


Theorem 18.1. Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $Z$ of $X$, one has $f(\overline{A}) \subset \overline{f(A)}$.
3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

Proof. (1)$\Rightarrow$(2) Suppose $f$ is continuous. Let $A \subset X$ and $x \in \overline{A}$. If $x \in A$ then $f(x) \in f(A) \subset \overline{f(A)}$. If $x \notin A$ then let $V$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open in $X$ and $x \in f^{-1}(V)$. By definition of $\overline{A}$, $f^{-1}(V)$ intersects $A$ at some point $y \neq x$. So $f(y) \in V \cap f(A)$ (notice that $f(y) \neq f(x)$ since $f(x) \notin f(A))$. So $f(x) \in \overline{f(A)}$. So $f(x) \in \overline{f(A)}$ for any $x \in \overline{A}$ and hence $f(\overline{A}) \subset \overline{f(A)}$. 


Theorem 18.1

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1. $f$ is continuous.
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3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

Proof. (1) $\Rightarrow$ (2) Suppose $f$ is continuous. Let $A \subset X$ and $x \in \overline{A}$. If $x \in A$ then $f(x) \in f(A) \subset f(\overline{A})$. If $x \notin A$ then let $V$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open in $X$ and $x \in f^{-1}(V)$. By definition of $\overline{A}$, $f^{-1}(V)$ intersects $A$ at some point $y \neq x$. So $f(y) \in V \cap f(A)$ (notice that $f(y) \neq f(x)$ since $f(x) \notin f(A)$). So $f(x) \in f(\overline{A})$. So $f(x) \in f(\overline{A})$ for any $x \in \overline{A}$ and hence $f(\overline{A}) \subset f(\overline{A})$. 


Theorem 18.1 (continued 1)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. let $f : X \rightarrow Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $Z$ of $X$, one has $f(A) \subset f(A)$.
3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

**Proof (continued).** (2)$\Rightarrow$(3) Let $B$ be closed in $Y$ and let $A = f^{-1}(B)$. Then $f(A) \subset B$ ($f$ may not be onto $B$ and so we may not have $f(A) = B$). So if $x \in \overline{A}$ then $f(x) \in f(A) \subset f(A)$ by hypothesis (2) and $f(A) \subset B = B$ since $f(A) \subset B$ and $B$ is closed.
Theorem 18.1 (continued 1)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $Z$ of $X$, one has $f(A) \subset f(A)$.
3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

**Proof (continued).** $(2) \Rightarrow (3)$ Let $B$ be closed in $Y$ and let $A = f^{-1}(B)$. Then $f(A) \subset B$ ($f$ may not be onto $B$ and so we may not have $f(A) = B$). So if $x \in A$ then $f(x) \in f(A) \subset f(A)$ by hypothesis (2) and $f(A) \subset B = B$ since $f(A) \subset B$ and $B$ is closed. Hence $f(x) \in B$ and $x \in f^{-1}(B) = A$. So $A \subset A$ and (since $A \subset A$) we have $A = \overline{A}$ so that $A = f^{-1}(B)$ is closed (by Lemma 17.A), as claimed.
Theorem 18.1. Let $X$ and $Y$ be topological spaces. let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $Z$ of $X$, one has $f(A) \subseteq f(A)$.
3. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subseteq V$.

Proof (continued). (2) $\Rightarrow$ (3) Let $B$ be closed in $Y$ and let $A = f^{-1}(B)$. Then $f(A) \subseteq B$ ($f$ may not be onto $B$ and so we may not have $f(A) = B$). So if $x \in \overline{A}$ then $f(x) \in f(A) \subseteq \overline{f(A)}$ by hypothesis (2) and $f(A) \subseteq B$ since $f(A) \subseteq B$ and $B$ is closed. Hence $f(x) \in B$ and $x \in f^{-1}(B) = A$. So $\overline{A} \subseteq A$ and (since $A \subseteq \overline{A}$) we have $A = \overline{A}$ so that $A = f^{-1}(B)$ is closed (by Lemma 17.A), as claimed.
Theorem 18.1 (continued 2)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. Let $f : X \rightarrow Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.

**Proof (continued).** $(3) \Rightarrow (1)$ Let $V$ be an open set in $Y$. Set $B = Y \setminus V$. Then

\[
f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1} \setminus f^{-1}(V) \text{ by Exercise 2.2(d)} = X \setminus f^{-1}(V) \text{ since } X \text{ is the domain of } f.
\]
Theorem 18.1 (continued 2)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.

**Proof (continued).** $(3) \Rightarrow (1)$ Let $V$ be an open set in $Y$. Set $B = Y \setminus V$. Then

$$f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1} \setminus f^{-1}(V) \text{ by Exercise 2.2(d)} = X \setminus f^{-1}(V) \text{ since } X \text{ is the domain of } f.$$

Since $V$ is open, $B$ is closed in $Y$ and so by hypothesis (3), $f^{-1}(B) = X \setminus f^{-1}(V)$ is closed in $X$ and so $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, $f$ is continuous.
Theorem 18.1. Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every closed subset $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.

Proof (continued). $(3) \Rightarrow (1)$ Let $V$ be an open set in $Y$. Set $B = Y \setminus V$. Then

$$f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1} \setminus f^{-1}(V) \text{ by Exercise 2.2(d)}$$

$$= X \setminus f^{-1}(V) \text{ since } X \text{ is the domain of } f.$$

Since $V$ is open, $B$ is closed in $Y$ and so by hypothesis (3), $f^{-1}(B) = X \setminus f^{-1}(V)$ is closed in $X$ and so $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, $f$ is continuous.
Theorem 18.1 (continued 3)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. let $f : X \to Y$. Then the following are equivalent:

(1) $f$ is continuous.

(4) For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

**Proof (continued).** $(1) \Rightarrow (4)$ Let $x \in X$ and let $V$ be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is open since $f$ is continuous and $x \in U$. That is, $f(U) \subset V$, as claimed.
Theorem 18.1 (continued 3)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.

4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subseteq V$.

**Proof (continued).** $(1) \Rightarrow (4)$ Let $x \in X$ and let $V$ be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is open since $f$ is continuous and $x \in U$. That is, $f(U) \subseteq V$, as claimed.

$(4) \Rightarrow (1)$ Let $V$ be an open set of $Y$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so by hypothesis (4) there is open $U_x$ in $X$ with $x \in U_x$ and $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$. 
Theorem 18.1 (continued 3)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.

4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

**Proof (continued).** (1) $\Rightarrow$ (4) Let $x \in X$ and let $V$ be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is open since $f$ is continuous and $x \in U$. That is, $f(U) \subset V$, as claimed.

(4) $\Rightarrow$ (1) Let $V$ be an open set of $Y$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so by hypothesis (4) there is open $U_x$ in $X$ with $x \in U_x$ and $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. Then with such open $U_x$ chosen for each $x \in f^{-1}(V)$ we have $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ and hence $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, $f$ is continuous and (1) follows.
Theorem 18.1 (continued 3)

**Theorem 18.1.** Let $X$ and $Y$ be topological spaces. let $f : X \to Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.

**Proof (continued).** (1)⇒(4) Let $x \in X$ and let $V$ be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is open since $f$ is continuous and $x \in U$. That is, $f(U) \subset V$, as claimed.

(4)⇒(1) Let $V$ be an open set of $Y$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so by hypothesis (4) there is open $U_x$ in $X$ with $x \in U_x$ and $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. Then with such open $U_x$ chosen for each $x \in f^{-1}(V)$ we have $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ and hence $f^{-1}(V)$ is open. Therefore, by the definition of continuous function, $f$ is continuous and (1) follows. □
Theorem 18.2. Rules for Constructing Continuous Functions.

Let $X$, $Y$, and $Z$ be topological spaces.

(a) (Constant Function) If $f : X \to Y$ maps all of $X$ into a single point $y_0 \in Y$, then $f$ is continuous.

(b) (Inclusion) if $A$ is a subspace of $X$, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let $V$ be open in $Y$. Then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \emptyset$ if $y_0 \notin V$. In either case, $f^{-1}(V)$ is open and so $f$ is continuous.
Theorem 18.2(a,b,c)

Theorem 18.2. Rules for Constructing Continuous Functions.
Let $X$, $Y$, and $Z$ be topological spaces.

(a) (Constant Function) If $f : X \rightarrow Y$ maps all of $X$ into a single point $y_0 \in Y$, then $f$ is continuous.

(b) (Inclusion) If $A$ is a subspace of $X$, the inclusion function $j : A \rightarrow X$ is continuous.

(c) (Composites) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the map $g \circ f : X \rightarrow Z$ is continuous.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let $V$ be open in $Y$. Then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \emptyset$ if $y_0 \notin V$. In either case, $f^{-1}(V)$ is open and so $f$ is continuous. (b) If $U$ is open in $X$, then $j^{-1}(U) = U \cap A$ which is open in $A$ (by definition of the subspace topology).
Theorem 18.2(a,b,c)

Theorem 18.2. Rules for Constructing Continuous Functions.
Let $X$, $Y$, and $Z$ be topological spaces.

(a) (Constant Function) If $f : X \rightarrow Y$ maps all of $X$ into a single point $y_0 \in Y$, then $f$ is continuous.

(b) (Inclusion) if $A$ is a subspace of $X$, the inclusion function $j : A \rightarrow X$ is continuous.

(c) (Composites) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the map $g \circ f : X \rightarrow Z$ is continuous.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let $V$ be open in $Y$. Then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \emptyset$ if $y_0 \notin V$. In either case, $f^{-1}(V)$ is open and so $f$ is continuous. (b) If $U$ is open in $X$, then $j^{-1}(U) = U \cap A$ which is open in $A$ (by definition of the subspace topology). (c) If $U$ is open in $Z$ then $g^{-1}(U)$ is open in $Y$ since $g$ is continuous and $f^{-1}(g^{-1}(U))$ is open in $X$ since $f$ is continuous. Now $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) = f^{-1}(g^{-1}(U))$ and so $g \circ f$ is continuous.
Theorem 18.2(a,b,c)

Theorem 18.2. Rules for Constructing Continuous Functions.
Let $X$, $Y$, and $Z$ be topological spaces.

(a) (Constant Function) If $f : X \to Y$ maps all of $X$ into a single point $y_0 \in Y$, then $f$ is continuous.

(b) (Inclusion) if $A$ is a subspace of $X$, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

Proof. (a) Let $f(x) = y_0$ for every $x \in X$. Let $V$ be open in $Y$. Then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \emptyset$ if $y_0 \notin V$. In either case, $f^{-1}(V)$ is open and so $f$ is continuous. (b) If $U$ is open in $X$, then $j^{-1}(U) = U \cap A$ which is open in $A$ (by definition of the subspace topology). (c) If $U$ is open in $Z$ then $g^{-1}(U)$ is open in $Y$ since $g$ is continuous and $f^{-1}(g^{-1}(U))$ is open in $X$ since $f$ is continuous. Now $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) = f^{-1}(g^{-1}(U))$ and so $g \circ f$ is continuous.
Theorem 18.2

Theorem 18.2. Rules for Constructing Continuous Functions.
Let $X$, $Y$, and $Z$ be topological spaces.

(d) (Restricting the Domain) If $f : X \to Y$ is continuous and if $A$ is a subspace of $X$, then the restricted function $f|_A : A \to Y$ is continuous.

(e) (Restricting or Expanding the Range) let $f : X \to Y$ be continuous. If $X$ is a subspace of $Y$ containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of $f$ is continuous. If $Z$ is a space having $Y$ as a subspace, then the functions $h : X \to Z$ obtained by expanding the range of $f$ is continuous.

(e) (Local Formulation of Continuity) The map $f : X \to Y$ is continuous if $X$ can be written as the union of open sets $U_\alpha$ such that $f|_{U_\alpha}$ is continuous for each $\alpha$. 
Theorem 18.2(d, e, f) (continued 1)

**Proof.** (d) The function $f|_A$ equals the composition of the inclusion map $j : A \to Y$ (which is continuous by part (b)) and $f : X \to Y$ (which is continuous by hypothesis). So by part (c), $f|_A$ is continuous.

(e) Let $f : X \to Y$ be continuous and $f(X) \subset Z \subset Y$. Let $B$ be open in $Z$. Then (by definition) $B = Z \cap U$ for some open $U$ in $Y$. 
Theorem 18.2(d, e, f) (continued 1)

**Proof.** (d) The function $f|_A$ equals the composition of the inclusion map $j : A \to Y$ (which is continuous by part (b)) and $f : X \to Y$ (which is continuous by hypothesis). So by part (c), $f|_A$ is continuous.

(e) Let $f : X \to Y$ be continuous and $f(X) \subset Z \subset Y$. Let $B$ be open in $Z$. Then (by definition) $B = Z \cap U$ for some open $U$ in $Y$. Then

$$g^{-1}(B) = g^{-1}(Z \cap U) = g^{-1}(Z) \cap g^{-1}(U)$$

$$= X \cap g^{-1}(U)$$

since $f(X) = g(X) \subset Z$

$$= g^{-1}(U)$$

$$= f^{-1}(U)$$

since $f(x) \in Y$ for some $x \in X$ implies $g(x) = f(x) \in U$.

Since $f$ is continuous, $f^{-1}(U)$ is open in $X$ and so $g^{-1}(U)$ is open in $X$. Therefore, $g$ is continuous.
Theorem 18.2(d, e, f) (continued 1)

Proof. (d) The function $f|_A$ equals the composition of the inclusion map $j : A \rightarrow Y$ (which is continuous by part (b)) and $f : X \rightarrow Y$ (which is continuous by hypothesis). So by part (c), $f|_A$ is continuous.

(e) Let $f : X \rightarrow Y$ be continuous and $f(X) \subset Z \subset Y$. Let $B$ be open in $Z$. Then (by definition) $B = Z \cap U$ for some open $U$ in $Y$. Then

$$g^{-1}(B) = g^{-1}(Z \cap U) = g^{-1}(Z) \cap g^{-1}(U)$$

$$= X \cap g^{-1}(U) \text{ since } f(X) = g(X) \subset Z$$

$$= g^{-1}(U)$$

$$= f^{-1}(U) \text{ since } f(x) \in Y \text{ for some } x \in X \text{ implies } g(x) = f(x) \in$$

Since $f$ is continuous, $f^{-1}(U)$ is open in $X$ and so $g^{-1}(U)$ is open in $X$. Therefore, $g$ is continuous.

Now let $h : X \rightarrow Z \supset Y$ be as described. Then $h$ is the composition of $f : X \times Y$ (which is continuous by hypothesis) and the inclusion map $j : Y \rightarrow Z$ (which is continuous by part (b)). So, by part (c), $h$ is continuous.
**Theorem 18.2(d, e, f) (continued 1)**

**Proof.** (d) The function $f|_A$ equals the composition of the inclusion map $j : A \to Y$ (which is continuous by part (b)) and $f : X \to Y$ (which is continuous by hypothesis). So by part (c), $f|_A$ is continuous.

(e) Let $f : X \to Y$ be continuous and $f(X) \subset Z \subset Y$. Let $B$ be open in $Z$. Then (by definition) $B = Z \cap U$ for some open $U$ in $Y$. Then

$$g^{-1}(B) = \quad g^{-1}(Z \cap U) = g^{-1}(Z) \cap g^{-1}(U)$$

$$= \quad X \cap g^{-1}(U) \text{since } f(X) = g(X) \subset Z$$

$$= \quad g^{-1}(U)$$

$$= \quad f^{-1}(U) \text{ since } f(x) \in Y \text{ for some } x \in X \text{ implies } g(x) = f(x) \in U.$$

Since $f$ is continuous, $f^{-1}(U)$ is open in $X$ and so $g^{-1}(U)$ is open in $X$. Therefore, $g$ is continuous.

Now let $h : X \to Z \supset Y$ be as described. Then $h$ is the composition of $f : X \times Y$ (which is continuous by hypothesis) and the inclusion map $j : Y \to Z$ (which is continuous by part (b)). So, by part (c), $h$ is continuous.
Theorem 18.2(d, e, f) (continued 2)

**Proof.** (f) Suppose \( X = \bigcup_{\alpha \in J} U_\alpha \) for open \( U_\alpha \) in \( X \) where \( f|_{U_\alpha} \) is continuous for each \( \alpha \in J \). Let \( V \) be an open set in \( Y \). Since \( f^{-1}(V) \cap U_\alpha \) consists of \( x \in X \cap U_\alpha = U_\alpha \) such that \( f(x) \in V \) and \( (f|_{U_\alpha})^{-1}(V) \) consists of \( x \in U_\alpha \) such that \( f(x) \in U_\alpha \), then \( f^{-1}(V) \cap U_\alpha = (f|_{U_\alpha})^{-1}(V) \) for all \( \alpha \in J \). Since \( f|_{U_\alpha} \) is continuous by hypothesis, then this set is open in \( U_\alpha \) and since \( U_\alpha \) is open then (by Lemma 16.2) this set is open in \( X \).
Theorem 18.2(d, e, f) (continued 2)

Proof. (f) Suppose $X = \bigcup_{\alpha \in J} U_\alpha$ for open $U_\alpha$ in $X$ where $f|_{U_\alpha}$ is continuous for each $\alpha \in J$. Let $V$ be an open set in $Y$. Since $f^{-1}(V) \cap U_\alpha$ consists of $x \in X \cap U_\alpha = U_\alpha$ such that $f(x) \in V$ and $(f|_{U_\alpha})^{-1}(V)$ consists of $x \in U_\alpha$ such that $f(x) \in U_\alpha$, then $f^{-1}(V) \cap U_\alpha = (f|_{U_\alpha})^{-1}(V)$ for all $\alpha \in J$. Since $f|_{U_\alpha}$ is continuous by hypothesis, then this set is open in $U_\alpha$ and since $U_\alpha$ is open then (by Lemma 16.2) this set is open in $X$. Since $X = \bigcup_{\alpha \in J} U_\alpha$ then

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap (\bigcup_{\alpha \in J} U_\alpha) = \bigcup_{\alpha \in J}(f^{-1}(V) \cap U_\alpha)$$

is open in $X$ since each set in the union is open. Therefore (by definition) $f$ is continuous. \qed
Theorem 18.2(d, e, f) (continued 2)

Proof. (f) Suppose \( X = \bigcup_{\alpha \in J} U_{\alpha} \) for open \( U_{\alpha} \) in \( X \) where \( f|_{U_{\alpha}} \) is continuous for each \( \alpha \in J \). Let \( V \) be an open set in \( Y \). Since \( f^{-1}(V) \cap U_{\alpha} \) consists of \( x \in X \cap U_{\alpha} = U_{\alpha} \) such that \( f(x) \in V \) and \((f|_{U_{\alpha}})^{-1}(V)\) consists of \( x \in U_{\alpha} \) such that \( f(x) \in U_{\alpha} \), then \( f^{-1}(V) \cap U_{\alpha} = (f|_{U_{\alpha}})^{-1}(V) \) for all \( \alpha \in J \). Since \( f|_{U_{\alpha}} \) is continuous by hypothesis, then this set is open in \( U_{\alpha} \) and since \( U_{\alpha} \) is open then (by Lemma 16.2) this set is open in \( X \). Since \( X = \bigcup_{\alpha \in J} U_{\alpha} \) then

\[
f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap (\bigcup_{\alpha \in J} U_{\alpha}) = \bigcup_{\alpha \in J} (f^{-1}(V) \cap U_{\alpha})
\]

is open in \( X \) since each set in the union is open. Therefore (by definition) \( f \) is continuous. \( \square \)
Theorem 18.3. The Pasting Lemma for Closed Sets.
Let \( X = A \cup B \) where \( A \) and \( B \) are closed in \( X \). Let \( f : A \rightarrow Y \) and \( g : B \rightarrow Y \) be continuous. If \( f(x) = g(x) \) for all \( x \in A \cup B \), then \( f \) and \( g \) combine (or “paste”) to give a continuous function \( h : X \rightarrow Y \) defined by setting \( h(x) = f(x) \) if \( x \in A \) and \( h(x) = g(x) \) if \( x \in B \).

Proof. Let \( C \) be closed in \( Y \). Then \( h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \).
Theorem 18.3. The Pasting Lemma for Closed Sets.

Let \( X = A \cup B \) where \( A \) and \( B \) are closed in \( X \). Let \( f : A \to Y \) and \( g : B \to Y \) be continuous. If \( f(x) = g(x) \) for all \( x \in A \cup B \), then \( f \) and \( g \) combine (or “paste”) to give a continuous function \( h : X \to Y \) defined by setting \( h(x) = f(x) \) if \( x \in A \) and \( h(x) = g(x) \) if \( x \in B \).

Proof. Let \( C \) be closed in \( Y \). Then \( h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \). Since \( f \) is continuous by hypothesis then \( f^{-1}(C) \) is closed in \( A \), by Theorem 18.1 (the (1) \( \Rightarrow \) (3) part), and so \( f^{-1}(C) \) is closed in \( X \) since \( A \) is closed (that is, \( f^{-1}(C) = A \cap D \) for closed \( D \) in \( X \), so \( f^{-1}(C) \) is closed in \( X \)). Similarly, \( g^{-1}(C) \) is closed in \( B \) and in \( X \).
Theorem 18.3. The Pasting Lemma for Closed Sets.
Let $X = A \cup B$ where $A$ and $B$ are closed in $X$. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cup B$, then $f$ and $g$ combine (or “paste”) to give a continuous function $h : X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

**Proof.** Let $C$ be closed in $Y$. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since $f$ is continuous by hypothesis then $f^{-1}(C)$ is closed in $A$, by Theorem 18.1 (the $(1) \Rightarrow (3)$ part), and so $f^{-1}(C)$ is closed in $X$ since $A$ is closed (that is, $f^{-1}(C) = A \cap D$ for closed $D$ in $X$, so $f^{-1}(C)$ is closed in $X$).

Similarly, $g^{-1}(C)$ is closed in $B$ and in $X$. Therefore $h^{-1}(C)$ is closed in $X$ and so by Theorem 18.2 (the $(3) \Rightarrow (1)$ part) $h$ is continuous. 

\[ \Box \]
Theorem 18.3. The Pasting Lemma for Closed Sets.

Let $X = A \cup B$ where $A$ and $B$ are closed in $X$. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cup B$, then $f$ and $g$ combine (or “paste”) to give a continuous function $h : X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

**Proof.** Let $C$ be closed in $Y$. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since $f$ is continuous by hypothesis then $f^{-1}(C)$ is closed in $A$, by Theorem 18.1 (the $(1) \Rightarrow (3)$ part), and so $f^{-1}(C)$ is closed in $X$ since $A$ is closed (that is, $f^{-1}(C) = A \cap D$ for closed $D$ in $X$, so $f^{-1}(C)$ is closed in $X$). Similarly, $g^{-1}(C)$ is closed in $B$ and in $X$. Therefore $h^{-1}(C)$ is closed in $X$ and so by Theorem 18.2 (the $(3) \Rightarrow (1)$ part) $h$ is continuous. \qed
Theorem 18.4. Maps into Products.

Let \( f : A \to X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \to X \) and \( f_2 : Y \to B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

**Proof.** Let \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \). Then for \( U \) open in \( X \) and \( V \) open in \( Y \), we have \( \pi_1^{-1}(U) = U \times T \) and \( \pi_2^{-1}(V) = X \times V \) open in \( X \times Y \) (by the definition of product topology; these are basis elements for the product topology on \( X \times Y \)). So \( \pi_1 \) and \( \pi_2 \) are continuous.
Theorem 18.4. Maps into Products.

Let $f : A \rightarrow X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \rightarrow X$ and $f_2 : Y \rightarrow B$. Then $f$ is continuous if and only if the functions $f_1$ and $f_2$ are continuous.

Proof. Let $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$. Then for $U$ open in $X$ and $V$ open in $Y$, we have $\pi_1^{-1}(U) = U \times T$ and $\pi_2^{-1}(V) = X \times V$ open in $X \times Y$ (by the definition of product topology; these are basis elements for the product topology on $X \times Y$). So $\pi_1$ and $\pi_2$ are continuous. Note that for each $a \in A$, $\pi_1(f(a)) = \pi_1((f_1(a), f_2(a))) = f_1(a)$ and $\pi_2(f(a)) = \pi_2((f_1(a), f_2(a))) = f_2(a)$. So $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. 
Theorem 18.4. Maps into Products.

Let \( f : A \to X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \to X \) and \( f_2 : Y \to B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

Proof. Let \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \). Then for \( U \) open in \( X \) and \( V \) open in \( Y \), we have \( \pi_1^{-1}(U) = U \times T \) and \( \pi_2^{-1}(V) = X \times V \) open in \( X \times Y \) (by the definition of product topology; these are basis elements for the product topology on \( X \times Y \)). So \( \pi_1 \) and \( \pi_2 \) are continuous. Note that for each \( a \in A \), \( \pi_1(f(a)) = \pi_1((f_1(a), f_2(a)) = f_1(a) \) and \( \pi_2(f(a)) = \pi_2((f_1(a), f_2(a)) = f_2(a) \). So \( f_1 = \pi_1 \circ f \) and \( f_2 = \pi_2 \circ f \).

Suppose \( f \) is continuous. Then, by Theorem 18.2 part (c), \( f_1 \) and \( f_2 \) are continuous.
Theorem 18.4. Maps into Products.
Let \( f : A \rightarrow X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \rightarrow X \) and \( f_2 : Y \rightarrow B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

Proof. Let \( \pi_1 : X \times Y \rightarrow X \) and \( \pi_2 : X \times Y \rightarrow Y \). Then for \( U \) open in \( X \) and \( V \) open in \( Y \), we have \( \pi_1^{-1}(U) = U \times T \) and \( \pi_2^{-1}(V) = X \times V \) open in \( X \times Y \) (by the definition of product topology; these are basis elements for the product topology on \( X \times Y \)). So \( \pi_1 \) and \( \pi_2 \) are continuous. Note that for each \( a \in A \), \( \pi_1(f(a)) = \pi_1((f_1(a), f_2(a)) = f_1(a) \) and \( \pi_2(f(a)) = \pi_2((f_1(a), f_2(a)) = f_2(a) \). So \( f_1 = \pi_1 \circ f \) and \( f_2 = \pi_2 \circ f \).

Suppose \( f \) is continuous. Then, by Theorem 18.2 part (c), \( f_1 \) and \( f_2 \) are continuous.
Theorem 18.4 (continued)

**Theorem 18.4. Maps into Products.**
Let \( f : A \to X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \to X \) and \( f_2 : Y \to B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

**Proof (continued).** Suppose \( f_1 \) and \( f_2 \) are continuous. Let \( U \times V \) be a basis element for the product topology of \( X \times Y \) (so \( U \) is open in \( X \) and \( V \) is open in \( Y \)).
Theorem 18.4 (continued)

**Theorem 18.4. Maps into Products.** Let \( f : A \to X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \to X \) and \( f_2 : Y \to B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

**Proof (continued).** Suppose \( f_1 \) and \( f_2 \) are continuous. Let \( U \times V \) be a basis element for the product topology of \( X \times Y \) (so \( U \) is open in \( X \) and \( V \) is open in \( Y \)). Now \( a \in f^{-1}(U \times V) \) if and only if \( f(a) \in U \times V \), or if and only if \( f_1(a) \in U \) and \( f_2(a) \in V \), or if and only if \( a \in f_1^{-1}(U) \cap f_2^{-1}(V) \). That is, \( f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V) \). Since \( f_1 \) and \( f_2 \) are continuous then \( f_1^{-1}(U) \) and \( f_2^{-1}(V) \) are open in \( X \) and so \( f^{-1}(U \times V) \) is open in \( X \).
Theorem 18.4. Maps into Products.
Let \( f : A \to X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \to X \) and \( f_2 : Y \to B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

Proof (continued). Suppose \( f_1 \) and \( f_2 \) are continuous. Let \( U \times V \) be a basis element for the product topology of \( X \times Y \) (so \( U \) is open in \( X \) and \( V \) is open in \( Y \)). Now \( a \in f^{-1}(U \times V) \) if and only if \( f(a) \in U \times V \), or if and only if \( f_1(a) \in U \) and \( f_2(a) \in V \), or if and only if \( a \in f_1^{-1}(U) \cap f_2^{-1}(V) \). That is, \( f^{-1}(U \times V) = f_1(U) \cap f_2^{-1}(V) \). Since \( f_1 \) and \( f_2 \) are continuous then \( f_1^{-1}(U) \) and \( f_2^{-1}(V) \) are open in \( X \) and so \( f^{-1}(U \times V) \) is open in \( X \). Since every open set in \( X \times Y \) can be written as a union of basis elements by Lemma 13.1, say \( \bigcup_{\alpha \in J} U_\alpha \times V_\alpha \), and \( f^{-1}(\bigcup_{\alpha \in J} U_\alpha \times V_\alpha) = \bigcup_{\alpha \in J} f^{-1}(U_\alpha \times V_\alpha) \), then the inverse image of any open set in \( X \times Y \) is open in \( A \). That is, \( f \) is continuous.
Theorem 18.4 (continued)

**Theorem 18.4. Maps into Products.** Let \( f : A \to X \times Y \) be given by the equation \( f(a) = (f_1(a), f_2(a)) \) where \( f_1 : A \to X \) and \( f_2 : Y \to B \). Then \( f \) is continuous if and only if the functions \( f_1 \) and \( f_2 \) are continuous.

**Proof (continued).** Suppose \( f_1 \) and \( f_2 \) are continuous. Let \( U \times V \) be a basis element for the product topology of \( X \times Y \) (so \( U \) is open in \( X \) and \( V \) is open in \( Y \)). Now \( a \in f^{-1}(U \times V) \) if and only if \( f(a) \in U \times V \), or if and only if \( f_1(a) \in U \) and \( f_2(a) \in V \), or if and only if \( a \in f_1^{-1}(U) \cap f_2^{-1}(V) \). That is, \( f^{-1}(U \times V) = f_1(U) \cap f_2^{-1}(V) \). Since \( f_1 \) and \( f_2 \) are continuous then \( f_1^{-1}(U) \) and \( f_2^{-1}(V) \) are open in \( X \) and so \( f^{-1}(U \times V) \) is open in \( A \). Since every open set in \( X \times Y \) can be written as a union of basis elements by Lemma 13.1, say \( \bigcup_{\alpha \in J} U_\alpha \times V_\alpha \), and \( f^{-1}(\bigcup_{\alpha \in J} U_\alpha \times V_\alpha) = \bigcup_{\alpha \in J} f^{-1}(U_\alpha \times V_\alpha) \), then the inverse image of any open set in \( X \times Y \) is open in \( A \). That is, \( f \) is continuous.