Chapter 2. Topological Spaces and Continuous Functions
Section 21. The Metric Topology (Continued)—Proofs of Theorems
Theorem 21.1

**Theorem 21.1.** Let $f : X \to Y$. Let $X$ and $Y$ be metrizable with metrics $d_X$ and $d_Y$, respectively. Then continuity of $f$ is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$ 

**Proof.** Suppose $f$ is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Consider the set $f^{-1}(B(f(x), \varepsilon))$. 


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$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$

Proof. Suppose $f$ is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Consider the set $f^{-1}(B(f(x), \varepsilon))$. Since $f$ is continuous then by definition of continuity, $f^{-1}(B(f(x), \varepsilon))$ is open in $X$ since $B(f(x), \varepsilon)$ is open and $dx \in f^{-1}(B(f(x), \varepsilon))$ then by Lemma 20.A there is $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. 


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$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$ 

Proof. Suppose $f$ is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Consider the set $f^{-1}(B(f(x), \varepsilon))$. Since $f$ is continuous then by definition of continuity, $f^{-1}(B(f(x), \varepsilon))$ is open in $X$ since $B(f(x), \varepsilon)$ is open and $x \in f^{-1}(B(f(x), \varepsilon))$ then by Lemma 20.A there is $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Then $d_X(x, y) < \delta$ implies $y \in B(f(x), \delta)$, so $f(y) \in B(f(x), \varepsilon)$ and $d_Y(f(x), f(y)) < \varepsilon$. 
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**Proof.** Suppose $f$ is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Consider the set $f^{-1}(B(f(x), \varepsilon))$. Since $f$ is continuous then by definition of continuity, $f^{-1}(B(f(x), \varepsilon))$ is open in $X$ since $B(f(x), \varepsilon)$ is open and $x \in f^{-1}(B(f(x), \varepsilon))$ then by Lemma 20.A there is $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Then $d_X(x, y) < \delta$ implies $y \in B(f(x), \delta)$, so $f(y) \in B(f(x), \varepsilon)$ and $d_Y(f(x), f(y)) < \varepsilon$. 

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$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$  

Proof (continued). Conversely suppose that the $\varepsilon/\delta$ condition is satisfied. Let $V$ be an open set in $Y$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$.  

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\]

Proof (continued). Conversely suppose that the \( \varepsilon/\delta \) condition is satisfied. Let \( V \) be an open set in \( Y \). Let \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \( V \) is open and \( df(x) \in V \) then by Lemma 20.B there is \( \varepsilon > 0 \) such that \( B(f(x), \varepsilon) \subset V \). By the \( \varepsilon/\delta \) hypothesis, there is \( \delta > 0 \) such that \( d_X(x, y) < \delta \) implies \( d_Y(f(x), f(y)) < \varepsilon \) (i.e., \( f(y) \in B(f(x), \varepsilon) \)).
Theorem 21.1 (continued)

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**Proof (continued).** Conversely suppose that the $\varepsilon/\delta$ condition is satisfied. Let $V$ be an open set in $Y$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since $V$ is open and $df(x) \in V$ then by Lemma 20.B there is $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset V$. By the $\varepsilon/\delta$ hypothesis, there is $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$ (i.e., $f(y) \in B(f(x), \varepsilon)$). So $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. So $x \in B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset V$. Therefore, by Lemma 20.B, $f^{-1}(V)$ is open and so $f$ is continuous. \qed
Theorem 21.1. Let $f : X \to Y$. Let $X$ and $Y$ be metrizable with metrics $d_X$ and $d_Y$, respectively. Then continuity of $f$ is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

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Lemma 21.2. The Sequence Lemma.
Let $X$ be a topological space. Let $A \subset X$. If there is a sequence of points of $A$ converging to $x$, then $x \in \overline{A}$. If $X$ is metrizable and $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \rightarrow x$.

Proof. Suppose that $\{x_n\} \rightarrow x$ where $\{x_n\} \subset A$. Then any given neighborhood $U$ of $x$, there is, by the definition of limit of a sequence (see Section 17) $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. 
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Proof (continued). Conversely, suppose that $X$ is metrizable and $x \in \overline{A}$. Let $d$ be a metric for the topology of $X$. For each $n \in \mathbb{N}$, consider $B_d(x, 1/n)$. This is an open set containing $x$ and so by Theorem 17.5 (part (a)), $B(x, 1/n)$ contains an element of $A$, say $x_n$. Then consider the resulting sequence $\{x_n\}$.
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Lemma 21.2 (continued)

Lemma 21.2. The Sequence Lemma.
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Theorem 21.3. Let $f : X \to Y$. If $f$ is continuous then for every convergent sequence $\{x_n\} \to x$ in $X$, the sequence $\{f(x_n)\} \to f(x)$ in $Y$. If $X$ is metrizable and for any sequence $\{x_n\} \to x$ in $X$ we have $\{f(x_n)\} \to f(x)$ in $Y$, then $f$ is continuous.

Proof. Suppose $f$ is continuous and let $\{x_n\} \to x$ in $X$. Let $V$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open and contains $x$. 
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Proof. Suppose $f$ is continuous and let $\{x_n\} \to x$ in $X$. Let $V$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is open and contains $x$. Since $\{x_n\} \to x$, by the definition of convergent sequence (see Section 17), there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in f^{-1}(V)$. Then $f(x_n) \in V$ for all $n \geq N$ and so (by definition again) $\{f(x_n)\} \to f(x)$. 
**Theorem 21.3.** Let \( f : X \to Y \). If \( f \) is continuous then for every convergent sequence \( \{x_n\} \to x \) in \( X \), the sequence \( \{f(x_n)\} \to f(x) \) in \( Y \). If \( X \) is metrizable and for any sequence \( \{x_n\} \to x \) in \( X \) we have \( \{f(x_n)\} \to f(x) \) in \( Y \), then \( f \) is continuous.

**Proof.** Suppose \( f \) is continuous and let \( \{x_n\} \to x \) in \( X \). Let \( V \) be a neighborhood of \( f(x) \). Then \( f^{-1}(V) \) is open and contains \( x \). Since \( \{x_n\} \to x \), by the definition of convergent sequence (see Section 17), there is \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have \( x_n \in f^{-1}(V) \). Then \( f(x_n) \in V \) for all \( n \geq N \) and so (by definition again) \( \{f(x_n)\} \to f(x) \).
Theorem 21.3 (continued)

**Theorem 21.3.** Let \( f : X \rightarrow Y \). If \( f \) is continuous then for every convergent sequence \( \{x_n\} \rightarrow x \) in \( X \), the sequence \( \{f(x_n)\} \rightarrow f(x) \) in \( Y \). If \( X \) is metrizable and for any sequence \( \{x_n\} \rightarrow x \) in \( X \) we have \( \{f(x_n)\} \rightarrow f(x) \) in \( Y \), then \( f \) is continuous.

**Proof (continued).** Conversely, suppose \( X \) is metrizable and suppose for any \( x \in X \) and any sequence \( \{x_n\} \rightarrow x \) in \( X \) we have \( \{f(x_n)\} \rightarrow f(x) \). Let \( A \subset X \). If \( x \in \overline{A} \) then there is a sequence \( \{x_n\} \subset A \) such that \( \{x_n\} \rightarrow x \) by Lemma 21.2 (part 2). By hypothesis, \( \{f(x_n)\} \rightarrow f(x) \). Since \( \{x_n\} \subset A \) then \( f(x_n) \in f(A) \) by Lemma 21.2 (part 1; notice that this does not require the metrizability of \( Y \)).
Theorem 21.3. Let \( f : X \rightarrow Y \). If \( f \) is continuous then for every convergent sequence \( \{x_n\} \rightarrow x \) in \( X \), the sequence \( \{f(x_n)\} \rightarrow f(x) \) in \( Y \). If \( X \) is metrizable and for any sequence \( \{x_n\} \rightarrow x \) in \( X \) we have \( \{f(x_n)\} \rightarrow f(x) \) in \( Y \), then \( f \) is continuous.

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Theorem 21.3. Let $f : X \to Y$. If $f$ is continuous then for every convergent sequence $\{x_n\} \to x$ in $X$, the sequence $\{f(x_n)\} \to f(x)$ in $Y$. If $X$ is metrizable and for any sequence $\{x_n\} \to x$ in $X$ we have $\{f(x_n)\} \to f(x)$ in $Y$, then $f$ is continuous.

Proof (continued). Conversely, suppose $X$ is metrizable and suppose for any $x \in X$ and any sequence $\{x_n\} \to x$ in $X$ we have $\{f(x_n)\} \to f(x)$. Let $A \subset X$. If $x \in \overline{A}$ then there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x$ by Lemma 21.2 (part 2). By hypothesis, $\{f(x_n)\} \to f(x)$. Since $\{x_n\} \subset A$ then $f(x_n) \in \overline{f(A)}$ by Lemma 21.2 (part 1; notice that this does not require the metrizability of $Y$). Since $x \in \overline{A}$ is arbitrary, then $f(\overline{A}) \subset \overline{f(A)}$. Hence, by Theorem 18.1 (the $(2) \Rightarrow (1)$ part), $f$ is continuous.
Theorem 21.5

Theorem 21.5. If $X$ is a topological space and if $f, g : X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all $x \in X$ then $f/g$ is continuous.

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ is continuous by Theorem 18.4 ("Maps Into Products"). The function $f + g$ equals the composition of $h$ and the addition operation $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Therefore $f + g$ is continuous by Theorem 18.2 part (c).
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Similarly, $f - g$ is the composition of $h$ and the subtraction operation $-: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f \cdot g$ is the composition of $h$ and the multiplication operation $\cdot: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $f/g$ is the composition of $h$ and the division operation $\div: \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$. So each of these is also continuous. □
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**Proof.** The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ is continuous by Theorem 18.4 (“Maps Into Products”). The function $f + g$ equals the composition of $h$ and the addition operation $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Therefore $f + g$ is continuous by Theorem 18.2 part (c).

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Theorem 21.6

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space $X$ to the metric space $Y$. If $\{f_n\}$ converges uniformly to $f$, then $f$ is continuous.

**Proof.** Let $V$ be open in $Y$ and let $x_0 \in f^{-1}(V)$. 


Theorem 21.6.

Uniform Limit Theorem.

Let \( f_n : X \to Y \) be a sequence of continuous functions from the topological space \( X \) to the metric space \( Y \). If \( \{f_n\} \) converges uniformly to \( f \), then \( f \) is continuous.

Proof. Let \( V \) be open in \( Y \) and let \( x_0 \in f^{-1}(V) \). Let \( y_0 = f(x_0) \in V \) and choose \( \varepsilon > 0 \) such that \( B(y_0, \varepsilon) \subset V \) (by Lemma 20.B). Since \( \{f_n\} \) converges uniformly to \( f \) on \( X \) then there is \( N \in \mathbb{N} \) such that for all \( n \geq N \) and for all \( x \in X \) we have

\[
(f_n(x), f(x)) < \varepsilon/3. \tag{*}
\]

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space $X$ to the metric space $Y$. If $\{f_n\}$ converges uniformly to $f$, then $f$ is continuous.

**Proof.** Let $V$ be open in $Y$ and let $x_0 \in f^{-1}(V)$. Let $y_0 = f(x_0) \in V$ and choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset V$ (by Lemma 20.B). Since $\{f_n\}$ converges uniformly to $f$ on $X$ then there is $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in X$ we have

$$(f_n(x), f(x)) < \varepsilon/3. \tag{\ast}$$

Since $f_N$ is continuous, there is a neighborhood $U$ of $x_0$ such that

$$f_N(U) \subset B(f_N(x_0), \varepsilon/3) \tag{\ast\ast}$$

by Theorem 18.1 (the $(1) \implies (4)$ part where $B(f_N(x_0), \varepsilon/3)$ is treated as a neighborhood of $f(x_0)$).

Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from the topological space $X$ to the metric space $Y$. If $\{f_n\}$ converges uniformly to $f$, then $f$ is continuous.

**Proof.** Let $V$ be open in $Y$ and let $x_0 \in f^{-1}(V)$. Let $y_0 = f(x_0) \in V$ and choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset V$ (by Lemma 20.B). Since $\{f_n\}$ converges uniformly to $f$ on $X$ then there is $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in X$ we have

$$ (f_n(x), f(x)) < \varepsilon/3. \quad (*) $$

Since $f_N$ is continuous, there is a neighborhood $U$ of $x_0$ such that

$$ f_N(U) \subset B(f_N(x_0), \varepsilon/3) \quad (**), $$

by Theorem 18.1 (the (1)$\Rightarrow$(4) part where $B(f_N(x_0), \varepsilon/3)$ is treated as a neighborhood of $f(x_0)$).
Proof (continued). Next, if \( x \in U \) then

\[
d(f(x), f_N(x)) < \varepsilon/3 \quad \text{by (\( \star \)) with } n = N
\]

\[
d(f_N(x), f_N(x_0)) < \varepsilon/3 \quad \text{by (\( \star\star \)) since } x \in U
\]

\[
d(f_N(x_0), f(x_0)) < \varepsilon/3 \quad \text{by (\( \star \)) with } n = N \text{ and } x = x_0.
\]

Then by the Triangle Inequality,

\[
d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

for all \( x \in U \).
Theorem 21.6 (continued)

Proof (continued). Next, if \( x \in U \) then

\[
d(f(x), f_N(x)) < \varepsilon/3 \text{ by } \ast \text{ with } n = N
\]

\[
d(f_N(x), f_N(x_0)) < \varepsilon/3 \text{ by } \ast \ast \text{ since } x \in U
\]

\[
d(f_N(x_0), f(x_0)) < \varepsilon/3 \text{ by } \ast \text{ with } n = N \text{ and } x = x_0.
\]

Then by the Triangle Inequality,

\[
d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

for all \( x \in U \). So \( U \) is a neighborhood of \( x_0 \) with \( f(U) \subset B(f(x_0), \varepsilon) \subset V \). So by Theorem 18.1 (the (4)⇒(1) part), \( f \) is continuous. \( \square \)
**Theorem 21.6 (continued)**

**Proof (continued).** Next, if \( x \in U \) then

\[
d(f(x), f_N(x)) < \frac{\varepsilon}{3} \text{ by (\ast) with } n = N
\]

\[
d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3} \text{ by (\ast\ast) since } x \in U
\]

\[
d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} \text{ by (\ast) with } n = N \text{ and } x = x_0.
\]

Then by the Triangle Inequality,

\[
d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

for all \( x \in U \). So \( U \) is a neighborhood of \( x_0 \) with \( f(U) \subset B(f(x_0), \varepsilon) \subset V \). So by Theorem 18.1 (the (4)\( \Rightarrow \)(1) part), \( f \) is continuous. \( \square \)