Theorem 22.1

Similarly, saturated closed sets of \( \mathcal{A} \) to closed sets of \( \mathcal{A} \) and \( \lambda \) is continuous and maps saturated open sets to open sets of \( \mathcal{A} \).

Proof. Suppose \( d \) is a quotient map. Then \( d \) is continuous (since the

Lemma 22.2.4

Lemma 22.4 (continued)

\[ d^{-1}(d^{-1}(n)) = (d^{-1}(n))^{-1} = n \]

Lemma 22.4

Chapter 2. Topological Spaces and Continuous Functions

Section 22. The Quotient Topology—Proofs of Theorems

Introduction to Topology
Theorem 2.2 (continued)

Proof. For each $y \in Y$, the set $\mathcal{A}_d = \{(x, y) \in X \times Y \mid d(x, y) \leq \epsilon\}$ is a one-point set in $Z$. Since $Z$ is a quotient map, the composition is continuous.

If $f$ is continuous, then the composition $d \circ f = g$ is continuous, by Definition.

Suppose $g$ is continuous, then the composition $f = d \circ g$ is continuous, since $d$ is continuous.

If $f$ is continuous and only if $g$ is continuous, then $f$ is continuous if and only if $g$ is continuous, since $Z$ is open and hence $f$ is continuous.

Thus $f$ is continuous if and only if $g$ is continuous.

Proof. Let $d$ be a quotient map. Let $Z \leftarrow X \times Y$ be a map that is constant on each set $\mathcal{A}_d$.

Theorem 2.2. Let $d$ be a quotient map. Let $Z \leftarrow X \times Y$ be a map that is constant on each set $\mathcal{A}_d$.

The proof of this theorem follows similar steps and reasoning as in the previous theorem.
The two given points of $X$. Hence $X$ is Hausdorff. Suppose $Z$ is Hausdorff. For distinct elements of $X$, their images under $f$ are disjoint. Since $f$ is continuous, $f$ is a quotient map. Therefore, $f$ is a quotient map. Since $f$ is open, so $f$ is a quotient map. Now $f$ is a quotient map by definition (see the definition of "quotient topology". So the composition $g = f \circ d$ is a quotient map by definition of $Y$. Since $f$ is open and $Z$ is open, $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open sets. Thus $f$ is a homeomorphism. Therefore, $X$ is Hausdorff.

**Corollary 2.2.3 (continued)**

$X$ is Hausdorff, so $f(X)$ is Hausdorff. Let $Z : X \rightarrow f(X)$ be the projection map that carries each point in $X$ to the element of $f(X)$. By Theorem 2.2.2, since $f$ is a quotient map, $Z$ is continuous.

**Corollary 2.2.3**

Let $\mathcal{T} \subseteq \mathcal{P}(Z)$ be the following collection of subsets of $X$: $\{Z \subseteq X \mid \{z\}_{T} = \emptyset \}$. Let $\mathcal{Y} = \{Z \subseteq X \mid \{z\}_{T} \neq \emptyset \}$.

**Proof** (continued) Suppose that $f$ is a quotient map. Then, by the definition of quotient topology, $f$ is a quotient map. Therefore, $f$ is a quotient map. Since $f$ is open, so $f$ is a quotient map. Now $f$ is a quotient map by definition (see the definition of "quotient topology". So the composition $g = f \circ d$ is a quotient map by definition of $Y$. Since $f$ is open and $Z$ is open, $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open sets. Thus $f$ is a homeomorphism. Therefore, $X$ is Hausdorff.

**Theorem 2.2.2 (continued)**

Let $\mathcal{T} \subseteq \mathcal{P}(Z)$ be the following collection of subsets of $X$: $\{Z \subseteq X \mid \{z\}_{T} = \emptyset \}$. Let $\mathcal{Y} = \{Z \subseteq X \mid \{z\}_{T} \neq \emptyset \}$.

**Proof** (continued) Suppose that $f$ is a quotient map. Then, by the definition of quotient topology, $f$ is a quotient map. Therefore, $f$ is a quotient map. Since $f$ is open, so $f$ is a quotient map. Now $f$ is a quotient map by definition (see the definition of "quotient topology". So the composition $g = f \circ d$ is a quotient map by definition of $Y$. Since $f$ is open and $Z$ is open, $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open sets. Thus $f$ is a homeomorphism. Therefore, $X$ is Hausdorff.