Chapter 2. Topological Spaces and Continuous Functions
Section 22. The Quotient Topology—Proofs of Theorems
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Lemma 22.A. Let $X$ and $Y$ be topological spaces. Then $p : X \rightarrow Y$ is a quotient map if and only if $p$ is continuous and maps saturated open sets of $X$ to open sets of $Y$.

Proof. Suppose $p$ is a quotient map. Then $p$ is continuous (since the inverse image of every open set in $Y$ has an open inverse image in $X$, by definition of quotient map). Also, for any open saturated set $U \subset X$, there is open $A \subset Y$ with $p^{-1}(A) = U$. 
Lemma 22.A. Let $X$ and $Y$ be topological spaces. Then $p : X \to Y$ is a quotient map if and only if $p$ is continuous and maps saturated open sets of $X$ to open sets of $Y$.

Proof. Suppose $p$ is a quotient map. Then $p$ is continuous (since the inverse image of every open set in $Y$ has an open inverse image in $X$, by definition of quotient map). Also, for any open saturated set $U \subset X$, there is open $A \subset Y$ with $p^{-1}(A) = U$. Since $U = p(A)$ is open in $Y$ then (by definition of quotient map) $A$ is open in $X$. So if $p$ is a quotient map then $p$ is continuous and maps saturated open sets of $X$ to open sets of $Y$ (and similarly, saturated closed sets of $X$ to closed sets of $Y$).
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Lemma 22.A (continued)

**Lemma 22.A.** Let $X$ and $Y$ be topological spaces. Then $p : X \to Y$ is a quotient map if and only if $p$ is continuous and maps saturated open sets of $X$ to open sets of $Y$.

**Proof (continued).** Now suppose $p$ is continuous and maps saturated open sets of $X$ to open sets of $Y$. Since $p$ is continuous, then for any open $U \subset Y$ we have $p^{-1}(U)$ open in $X$. Now suppose $p^{-1}(U)$ is open in $X$. Then, by the not above, $p^{-1}(U)$ is saturated since it is the inverse image of some set in $Y$ (namely, $U$). Since $p^{-1}(U)$ is a saturated open set, we have hypothesized that $p(p^{-1}(U)) = U$ is open in $Y$. 
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Theorem 22.1

Theorem 22.1. Let $p : X \rightarrow Y$ be a quotient map. Let $A$ be a subspace of $X$ that is saturated with respect to $p$. Let $q : A \rightarrow p(A)$ be the map obtained by restricting $p$ to $S$, $q = p|_A$.

1. If $A$ is either open or closed in $X$, then $a$ is a quotient map.
2. If $p$ is either an open or a closed map, then $q$ is a quotient map.

Proof. **STEP 1.** Let $V \subset p(A)$. Then for each $v \in V$ there must be $a \in A$ such that $p(a) = v$. So $p^{-1}(\{v\}) \cap A$ includes $a$ and so is nonempty.
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\[
\text{if } V \subset p(A) \text{ then } q^{-1}(V) = p^{-1}(V).
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**Proof.** **STEP 1.** Let \( V \subset p(A) \). Then for each \( v \in V \) there must be \( a \in A \) such that \( p(a) = v \). So \( p^{-1}(\{v\}) \cap A \) includes \( a \) and so is nonempty. Since \( A \) is saturated with respect to \( p \), then \( p^{-1}(V) \subset A \). Since \( q = p|_A \) then \( q^{-1}(V) \) is all the points of \( A \) mapped by \( p \) into \( V \). That is,

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For any subsets \( U \subset X \) and \( A \subset X \) we have \( p(U \cap A) \subset p(U) \cap p(A) \) since \( U \cap A \subset U \) and \( U \cap A \subset A \). Suppose \( y = p(u) = p(a) \in p(U) \cap p(A) \) for \( u \in U \) and \( a \in A \). Since \( A \) is saturated with respect to \( p \) and \( p^{-1}(p(a)) \) includes \( a \in A \) (and so \( p^{-1}(p(a)) \cap A \neq \emptyset \)), then \( p^{-1}(V) \subset A \).
Theorem 22.1

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1. If \( A \) is either open or closed in \( X \), then \( a \) is a quotient map.
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**Proof.** **STEP 1.** Let \( V \subset p(A) \). Then for each \( v \in V \) there must be \( a \in A \) such that \( p(a) = v \). So \( p^{-1}(\{v\}) \cap A \) includes \( a \) and so is nonempty.

Since \( A \) is saturated with respect to \( p \), then \( p^{-1}(V) \subset A \). Since \( q = p|_A \) then \( q^{-1}(V) \) is all the points of \( A \) mapped by \( p \) into \( V \). That is,

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\text{if } V \subset p(A) \text{ then } q^{-1}(V) = p^{-1}(V).
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For any subsets \( U \subset X \) and \( A \subset X \) we have \( p(U \cap A) \subset p(U) \cap p(A) \) since \( U \cap A \subset U \) and \( U \cap A \subset A \). Suppose \( y = p(u) = p(a) \in p(U) \cap p(A) \) for \( u \in U \) and \( a \in A \). Since \( A \) is saturated with respect to \( p \) and \( p^{-1}(p(a)) \) includes \( a \in A \) (and so \( p^{-1}(p(a)) \cap A \neq \emptyset \)), then \( p^{-1}p(a) \subset A \).
Theorem 22.1 (continued 1)

**Proof (continued).** Since \( u \in p^{-1}(p(a)) \subset A \) then \( x \in U \cap A \). So \( y = p(u) \in p(U \cap A) \). So \( p(U) \cap p(A) \subset p(U \cap A) \). Therefore

\[
\text{if } U \subset X \text{ then } p(U \cap A) = p(U) \cap p(A).
\]
Theorem 22.1 (continued 1)

Proof (continued). Since $u \in p^{-1}(p(a)) \subset A$ then $x \in U \cap A$. So $y = p(u) \in p(U \cap A)$. So $p(U) \cap p(A) \subset p(U \cap A)$. Therefore

$$\text{if } U \subset X \text{ then } p(U \cap A) = p(U) \cap p(A).$$

STEP 2. Suppose set $A$ is open in $X$. Let $V \subset p(A)$ where $q^{-1}(V)$ is open in $A$. Since $q^{-1}(V)$ is open in $A$ and $A$ is open in $X$, then $q^{-1}(V)$ is open in $X$. Since $q^{-1}(V) = p^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in $X$. Since $p$ is a quotient map then (by definition) $V$ is open in $Y$. So $V$ is open in $p(A)$.
Proof (continued). Since $u \in p^{-1}(p(a)) \subset A$ then $x \in U \cap A$. So $y = p(u) \in p(U \cap A)$. So $p(U) \cap p(A) \subset p(U \cap A)$. Therefore

$$\text{if } U \subset X \text{ then } p(U \cap A) = p(U) \cap p(A).$$

STEP 2. Suppose set $A$ is open in $X$. Let $V \subset p(A)$ where $q^{-1}(V)$ is open in $A$. Since $q^{-1}(V)$ is open in $A$ and $A$ is open in $X$, then $q^{-1}(V)$ is open in $X$. Since $q^{-1}(V) = p^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in $X$. Since $p$ is a quotient map then (by definition) $V$ is open in $Y$. So $V$ is open in $p(A)$. So if $q^{-1}(V)$ is open in $A$ then $V$ is open in $p(A)$ (recall $q : A \rightarrow p(A)$). Since $p$ is a quotient map, then it is continuous (inverse images of open sets are open) and $q$ is a restriction of $p$, then $q$ is continuous (restrictions of continuous functions are continuous by Theorem 18.2(d)). So inverse images of open sets are open under $q$. Therefore, $q : A \rightarrow p(A)$ and $V$ is open in $p(A)$ if and only if $q^{-1}(V)$ is open in $A$. 
Theorem 22.1 (continued 1)

**Proof (continued).** Since \( u \in p^{-1}(p(a)) \subset A \) then \( x \in U \cap A \). So \( y = p(u) \in p(U \cap A) \). So \( p(U) \cap p(A) \subset p(U \cap A) \). Therefore

\[
\text{if } U \subset X \text{ then } p(U \cap A) = p(U) \cap p(A).
\]

**STEP 2.** Suppose set \( A \) is open in \( X \). Let \( V \subset p(A) \) where \( q^{-1}(V) \) is open in \( A \). Since \( q^{-1}(V) \) is open in \( A \) and \( A \) is open in \( X \), then \( q^{-1}(V) \) is open in \( X \). Since \( q^{-1}(V) = p^{-1}(V) \) by Step 1, then \( p^{-1}(V) \) is open in \( X \). Since \( p \) is a quotient map then (by definition) \( V \) is open in \( Y \). So \( V \) is open in \( p(A) \). So if \( q^{-1}(V) \) is open in \( A \) then \( V \) is open in \( p(A) \) (recall \( q : A \rightarrow p(A) \)). Since \( p \) is a quotient map, then it is continuous (inverse images of open sets are open) and \( q \) is a restriction of \( p \), then \( q \) is continuous (restrictions of continuous functions are continuous by Theorem 18.2(d)). So inverse images of open sets are open under \( q \). Therefore, \( q : A \rightarrow p(A) \) and \( V \) is open in \( p(A) \) if and only if \( q^{-1}(V) \) is open in \( A \).
Theorem 22.1 (continued 2)

**Proof (continued).** Since $p : X \to Y$ is surjective (onto) and $q = p|_A$, then $q : A \to p(A)$ is surjective. That is, $q$ is a quotient map, and (1) follows for $A$ open.

Suppose map $p$ is open. Let $A \subset p(A)$ where $q^{-1}(V)$ is open in $A$. Since $p^{-1}(V) = q^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in $A$. That is, $p^{-1}(V) = A \cap U$ for some open set $U$ in $X$. 
Theorem 22.1 (continued 2)

**Proof (continued).** Since $p : X \to Y$ is surjective (onto) and $q = p|_A$, then $q : A \to p(A)$ is surjective. That is, $q$ is a quotient map, and (1) follows for $A$ open.

Suppose map $p$ is open. Let $A \subset p(A)$ where $q^{-1}(V)$ is open in $A$. Since $p^{-1}(V) = q^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in $A$. That is, $p^{-1}(V) = A \cap U$ for some open set $U$ in $X$. Now $p(p^{-1}(V)) = V$ because $p$ is onto (surjective). Then $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$ by Step 1. Since $p$ is a quotient map and $U$ is open in $X$ then $p(U)$ is open in $Y$. Hence $V$ is open in $p(A)$. As in the previous paragraph, this is sufficient to show that $q$ is a quotient map and (2) follows for $p$ an open map.
Theorem 22.1 (continued 2)

Proof (continued). Since $p : X \to Y$ is surjective (onto) and $q = p|_A$, then $q : A \to p(A)$ is surjective. That is, $q$ is a quotient map, and (1) follows for $A$ open.

Suppose map $p$ is open. Let $A \subset p(A)$ where $q^{-1}(V)$ is open in $A$. Since $p^{-1}(V) = q^{-1}(V)$ by Step 1, then $p^{-1}(V)$ is open in $A$. That is, $p^{-1}(V) = A \cap U$ for some open set $U$ in $X$. Now $p(p^{-1}(V)) = V$ because $p$ is onto (surjective). Then $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$ by Step 1. Since $p$ is a quotient map and $U$ is open in $X$ then $p(U)$ is open in $Y$. Hence $V$ is open in $p(A)$. As in the previous paragraph, this is sufficient to show that $q$ is a quotient map and (2) follows for $p$ an open map.

STEP 3. The arguments in Step 2 follow through with “open” replace with “closed.” Therefore, (1) follows for set $A$ closed and (2) follows for map $p$ closed.
Theorem 22.1 (continued 2)

Proof (continued). Since \( p : X \to Y \) is surjective (onto) and \( q = p|_A \), then \( q : A \to p(A) \) is surjective. That is, \( q \) is a quotient map, and (1) follows for \( A \) open.

Suppose map \( p \) is open. Let \( A \subset p(A) \) where \( q^{-1}(V) \) is open in \( A \). Since \( p^{-1}(V) = q^{-1}(V) \) by Step 1, then \( p^{-1}(V) \) is open in \( A \). That is, \( p^{-1}(V) = A \cap U \) for some open set \( U \) in \( X \). Now \( p(p^{-1}(V)) = V \) because \( p \) is onto (surjective). Then \( V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A) \) by Step 1. Since \( p \) is a quotient map and \( U \) is open in \( X \) then \( p(U) \) is open in \( Y \). Hence \( V \) is open in \( p(A) \). As in the previous paragraph, this is sufficient to show that \( q \) is a quotient map and (2) follows for \( p \) an open map.

STEP 3. The arguments in Step 2 follow through with “open” replace with “closed.” Therefore, (1) follows for set \( A \) closed and (2) follows for map \( p \) closed.
Theorem 22.2. Let $p : X \to Y$ be a quotient map. Let $Z$ be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}\{y\}$, for $y \in Y$. Then $g$ induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map $f$ is continuous if and only if $g$ is continuous. $f$ is a quotient map if and only if $g$ is a quotient map.
Theorem 22.2 (continued 1)

**Proof.** For each $y \in Y$, the set $g(p^{-1}\{y\})$ is a one-point set in $Z$ since $g$ is constant on $p^{-1}\{y\}$. Define $f(y)$ to be this one point. Then $f : Y \to Z$ and for each $x \in W$ we have $f(p(x)) = g(x)$. So function $f$ exists as claimed.
Theorem 22.2 (continued 1)

Proof. For each \( y \in Y \), the set \( g(p^{-1}(\{y\})) \) is a one-point set in \( Z \) since \( g \) is constant on \( p^{-1}(\{y\}) \). Define \( f(y) \) to be this one point. Then \( f : Y \to Z \) and for each \( x \in W \) we have \( f(p(x)) = g(x) \). So function \( f \) exists as claimed.

If \( f \) is continuous, then the composition \( g = f \circ p \) is continuous (since \( p \) is a quotient map and so by definition is continuous).
Theorem 22.2 (continued 1)

**Proof.** For each \( y \in Y \), the set \( g(p^{-1}(\{y\})) \) is a one-point set in \( Z \) since \( g \) is constant on \( p^{-1}(\{y\}) \). Define \( f(y) \) to be this one point. Then \( f : Y \to Z \) and for each \( x \in W \) we have \( f(p(x)) = g(x) \). So function \( f \) exists as claimed.

If \( f \) is continuous, then the composition \( g = f \circ p \) is continuous (since \( p \) is a quotient map and so by definition is continuous).

Suppose \( g \) is continuous. Let \( V \) be an open set in \( Z \). Then \( g^{-1}(V) \) is open in \( X \). But \( g^{-1}(V) = p^{-1}(f^{-1}(V)) \) by above.
Proof. For each \( y \in Y \), the set \( g(p^{-1}(\{y\})) \) is a one-point set in \( Z \) since \( g \) is constant on \( p^{-1}(\{y\}) \). Define \( f(y) \) to be this one point. Then \( f : Y \to Z \) and for each \( x \in W \) we have \( f(p(x)) = g(x) \). So function \( f \) exists as claimed.

If \( f \) is continuous, then the composition \( g = f \circ p \) is continuous (since \( p \) is a quotient map and so by definition is continuous).

Suppose \( g \) is continuous. Let \( V \) be an open set in \( Z \). Then \( g^{-1}(V) \) is open in \( X \). But \( g^{-1}(V) = p^{-1}(f^{-1}(V)) \) by above. Since \( p \) is a quotient map, \( p^{-1}(f^{-1}(V)) \) is open if and only if \( f^{-1}(V) \) is open and hence, since \( p^{-1}(f^{-1}(V)) \) is open, then \( f^{-1}(V) \) is open and so \( f \) is continuous. So \( f \) is continuous if and only if \( g \) is continuous.
Theorem 22.2 (continued 1)

**Proof.** For each \( y \in Y \), the set \( g(p^{-1}({y})) \) is a one-point set in \( Z \) since \( g \) is constant on \( p^{-1}({y}) \). Define \( f(y) \) to be this one point. Then \( f : Y \to Z \) and for each \( x \in W \) we have \( f(p(x)) = g(x) \). So function \( f \) exists as claimed.

If \( f \) is continuous, then the composition \( g = f \circ p \) is continuous (since \( p \) is a quotient map and so by definition is continuous).

Suppose \( g \) is continuous. Let \( V \) be an open set in \( Z \). Then \( g^{-1}(V) \) is open in \( X \). But \( g^{-1}(V) = p^{-1}(f^{-1}(V)) \) by above. Since \( p \) is a quotient map, \( p^{-1}(f^{-1}(V)) \) is open if and only if \( f^{-1}(V) \) is open and hence, since \( p^{-1}(f^{-1}(V)) \) is open, then \( f^{-1}(V) \) is open and so \( f \) is continuous. So \( f \) is continuous if and only if \( g \) is continuous.

Suppose \( f \) is a quotient map. Then \( g \) is the composite of two quotient maps and hence is a quotient map (see page 141 for details).
Proof. For each $y \in Y$, the set $g(p^{-1}(\{y\}))$ is a one-point set in $Z$ since $g$ is constant on $p^{-1}(\{y\})$. Define $f(y)$ to be this one point. Then $f : Y \to Z$ and for each $x \in W$ we have $f(p(x)) = g(x)$. So function $f$ exists as claimed.

If $f$ is continuous, then the composition $g = f \circ p$ is continuous (since $p$ is a quotient map and so by definition is continuous).

Suppose $g$ is continuous. Let $V$ be an open set in $Z$. Then $g^{-1}(V)$ is open in $X$. But $g^{-1}(V) = p^{-1}(f^{-1}(V))$ by above. Since $p$ is a quotient map, $p^{-1}(f^{-1}(V))$ is open if and only if $f^{-1}(V)$ is open and hence, since $p^{-1}(f^{-1}(V))$ is open, then $f^{-1}(V)$ is open and so $f$ is continuous. So $f$ is continuous if and only if $g$ is continuous.

Suppose $f$ is a quotient map. Then $g$ is the composite of two quotient maps and hence is a quotient map (see page 141 for details).
Proof (continued). Suppose that $g$ is a quotient map. Then, by the definition of quotient map, $g$ is onto (surjective). Therefore $f$ is surjective.

Let $V \subseteq Z$ and suppose $f^{-1}(V)$ is open in $Y$. Then $p^{-1}(f^{-1}(V))$ is open in $X$ because $p$ is continuous. Since $g^{-1}(V) = p^{-1}(f^{-1}(V))$, then $g^{-1}(V)$ is open. Since $g$ is a quotient map, then $V$ is open in $Z$. 
Proof (continued). Suppose that $g$ is a quotient map. Then, by the definition of quotient map, $g$ is onto (surjective). Therefore $f$ is surjective. Let $V \subset Z$ and suppose $f^{-1}(V)$ is open in $Y$. Then $p^{-1}(f^{-1}(V))$ is open in $X$ because $p$ is continuous. Since $g^{-1}(V) = p^{-1}(f^{-1}(V))$, then $g^{-1}(V)$ is open. Since $g$ is a quotient map, then $V$ is open in $Z$. So if $f^{-1}(V)$ is open then $V$ is open. We have assumed that $f$ is a quotient map, so $g$ is continuous and by above, $f$ is continuous. So if $V$ is open in $Z$ then $f^{-1}(V)$ is open in $Y$. Therefore, $f$ is a quotient map. \[\Box\]
Theorem 22.2 (continued 2)

**Proof (continued).** Suppose that \( g \) is a quotient map. Then, by the definition of quotient map, \( g \) is onto (surjective). Therefore \( f \) is surjective. Let \( V \subset Z \) and suppose \( f^{-1}(V) \) is open in \( Y \). Then \( p^{-1}(f^{-1}(V)) \) is open in \( X \) because \( p \) is continuous. Since \( g^{-1}(V) = p^{-1}(f^{-1}(V)) \), then \( g^{-1}(V) \) is open. Since \( g \) is a quotient map, then \( V \) is open in \( Z \). So if \( f^{-1}(V) \) is open then \( V \) is open. We have assumed that \( f \) is a quotient map, so \( g \) is continuous and by above, \( f \) is continuous. So if \( V \) is open in \( Z \) then \( f^{-1}(V) \) is open in \( Y \). Therefore, \( f \) is a quotient map. \( \square \)
Corollary 22.3. Let \( g : X \to Z \) be a surjective continuous map. Let \( X^* \) be the following collection of subsets of \( X \): \( X^* = \{ g^{-1}\{z\} \mid z \in Z \} \). Let \( X^* \) have the quotient topology.

(a) The map \( g \) induces a bijective continuous map \( f : X^* \to Z \), which is a homeomorphism if and only if \( g \) is a quotient map.

(b) If \( Z \) is Hausdorff, so is \( X^* \).
Corollary 22.3 (continued 1)

**Proof.** Let \( p : X \to X^* \) be the projection map that carries each point in \( X \) to the element of \( X^* \) containing it. By Theorem 22.2, since \( g \) is hypothesized to be continuous, \( g \) induces a continuous map \( f : X^* \to Z \). As argued in the proof of Theorem 22.2, since \( f \circ p = g \) and \( g \) is surjective, then \( f \) is surjective. Suppose \( g^{-1}(\{z_1\}) = g^{-1}(\{z_2\}) \). Let \( x_1, x_2 \in X \) such that \( p(x_1) = g^{-1}(\{z_1\}) \) and \( p(x_2) = g^{-1}(\{z_2\}) \) (notice that projection \( p \) is onto \( X^* \)). So \( x_1 \in g^{-1}(\{z_1\}) \) and \( g^{-1}(\{z_2\}) \) must be disjoint (the \( g^{-1}(\{z\}) \)'s partition \( X \)). Hence \( z_1 \neq z_2 \) and \( x_1 \neq x_2 \) and so \( g(x_1) = z_1 \neq z_2 = g(x_2) \).
**Proof.** Let $p : X \rightarrow X^*$ be the projection map that carries each point in $X$ to the element of $X^*$ containing it. By Theorem 22.2, since $g$ is hypothesized to be continuous, $g$ induces a continuous map $f : X^* \rightarrow Z$. As argued in the proof of Theorem 22.2, since $f \circ p = g$ and $g$ is surjective, then $f$ is surjective. Suppose $g^{-1}(\{z_1\}) = g^{-1}(\{z_2\})$. Let $x_1, x_2 \in X$ such that $p(x_1) = g^{-1}(\{z_1\})$ and $p(x_2) = g^{-1}(\{z_2\})$ (notice that projection $p$ is onto $X^*$). So $x_1 \in g^{-1}(\{z_1\})$ and $g^{-1}(\{z_2\})$ must be disjoint (the $g^{-1}(\{z\})$’s partition $X$). Hence $z_1 \neq z_2$ and $x_1 \neq x_2$ and so $g(x_1) = z_1 \neq z_2 = g(x_2)$. So $(f \circ p)(x_1) = f(g^{-1}(\{z_1\})) = g(x_1) = z_1$ and $(f \circ p)(x_2) = f(g^{-1}(\{z_2\})) = g(x_2) = z_2$. That is, $f(g^{-1}(\{z_1\})) \neq f(g^{-1}(\{z_2\}))$, and so $f$ is one to one. So $f$ is a bijection.
Proof. Let \( p : X \to X^* \) be the projection map that carries each point in \( X \) to the element of \( X^* \) containing it. By Theorem 22.2, since \( g \) is hypothesized to be continuous, \( g \) induces a continuous map \( f : X^* \to Z \). As argued in the proof of Theorem 22.2, since \( f \circ p = g \) and \( g \) is surjective, then \( f \) is surjective. Suppose \( g^{-1}(\{z_1\}) = g^{-1}(\{z_2\}) \). Let \( x_1, x_2 \in X \) such that \( p(x_1) = g^{-1}(\{z_1\}) \) and \( p(x_2) = g^{-1}(\{z_2\}) \) (notice that projection \( p \) is onto \( X^* \)). So \( x_1 \in g^{-1}(\{z_1\}) \) and \( g^{-1}(\{z_2\}) \) must be disjoint (the \( g^{-1}(\{z\}) \)'s partition \( X \)). Hence \( z_1 \neq z_2 \) and \( x_1 \neq x_2 \) and so \( g(x_1) = z_1 \neq z_2 = g(x_2) \). So \( (f \circ p)(x_1) = f(g^{-1}(\{z_1\})) = g(x_1) = z_1 \) and \( (f \circ p)(x_2) = f(g^{-1}(\{z_2\})) = g(x_2) = z_2 \). That is, \( f(g^{-1}(\{z_1\})) \neq f(g^{-1}(\{z_2\})) \), and so \( f \) is one to one. So \( f \) is a bijection.
**Proof (continued).** Suppose $f$ is a homeomorphism. Then $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open. So $f$ is a quotient map. Now $p$ is a quotient map by definition (see the definition of “quotient topology”). So the composition $g = f \circ p$ is a quotient map. Then by Theorem 22.2, $f$ is a quotient map. Since $f$ is bijective as argued above, then $f$ is a homeomorphism. So (a) follows.
Corollary 22.3 (continued 2)

Proof (continued). Suppose $f$ is a homeomorphism. Then $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open. So $f$ is a quotient map. Now $p$ is a quotient map by definition (see the definition of “quotient topology”). So the composition $g = f \circ p$ is a quotient map. Then by Theorem 22.2, $f$ is a quotient map. Since $f$ is bijective as argued above, then $f$ is a homeomorphism. So (a) follows.

Suppose $Z$ is Hausdorff. For distinct elements of $X^*$, their images under $f$ are distinct since $f$ is one to one by (a). So in $Z$ these images have disjoint neighborhoods $U$ and $V$. 
Corollary 22.3 (continued 2)

Proof (continued). Suppose $f$ is a homeomorphism. Then $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open. So $f$ is a quotient map. Now $p$ is a quotient map by definition (see the definition of “quotient topology”). So the composition $g = f \circ p$ is a quotient map. Then by Theorem 22.2, $f$ is a quotient map. Since $f$ is bijective as argued above, then $f$ is a homeomorphism. So (a) follows.

Suppose $Z$ is Hausdorff. For distinct elements of $X^*$, their images under $f$ are distinct since $f$ is one to one by (a). So in $Z$ these images have disjoint neighborhoods $U$ and $V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint ($f$ is a bijection) and open ($f$ is continuous by (a)) and are neighborhoods of the two given points of $X^*$. Hence $X^*$ is Hausdorff. □
**Corollary 22.3 (continued 2)**

**Proof (continued).** Suppose $f$ is a homeomorphism. Then $f$ maps open sets to open sets and since $f$ is continuous, inverse images of open sets are open. So $f$ is a quotient map. Now $p$ is a quotient map by definition (see the definition of “quotient topology”). So the composition $g = f \circ p$ is a quotient map. Then by Theorem 22.2, $f$ is a quotient map. Since $f$ is bijective as argued above, then $f$ is a homeomorphism. So (a) follows.

Suppose $Z$ is Hausdorff. For distinct elements of $X^*$, their images under $f$ are distinct since $f$ is one to one by (a). So in $Z$ these images have disjoint neighborhoods $U$ and $V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint ($f$ is a bijection) and open ($f$ is continuous by (a)) and are neighborhoods of the two given points of $X^*$. Hence $X^*$ is Hausdorff.