Notice that intervals and rays are convex sets and so are connected. We have shown that if \( Y \) is a convex subset of \( \mathbb{R} \) then \( Y \) is connected.

Fact: If \( c \in (a, b) \) there is \( \delta > 0 \) such that \( c \in (a, c + \delta) \) and \( c \in (c - \delta, b) \). Therefore, \( \delta \) is an upper bound of \( (a, b) \), \( c \in (a, b) \) is connected.

Proof: (continued)

Theorem 2.4.1. If \( L \) is a linear continuum in the order topology, then \( L \) is connected and so are intervals and rays in \( L \).

**Theorem 2.4.1 (continued I)**

**Theorem 2.4.1 (continued II)**

**Proof:** (continued)
Theorem 2.3. Intermediate Value Theorem

Let $f : X \rightarrow Y$ be a continuous map, where $X$ is a connected space and $Y$ is a
connected space from $A$ and $B$ is chosen from $Y$. But since $X$ is
connected and $f$ is continuous, then $f(A)$ is connected by Theorem 2.2.

Lemma 2.4. If spaces $X$ is path connected, then it is connected.

PROOF. Let $x$ be any path in $X$. Since $f$ is continuous and $[x, y]$ is a continuous set in $X$, so by theorem $2.3$, $f([x, y])$ is connected in $X$. So by Lemma 2.4, $f(A)$ is connected in $X$. Hence $f(A) = f([x, y])$ is connected in $X$. But since $X$ is
connected, $f(A)$ is also connected.

Theorem 2.4. A space $X$ is path connected if and only if for every $x \in X$ and $y \in X$, there exists a path $f : [x, y] \rightarrow X$.

Proof. Suppose $X$ and $Y$ are as hypothesized. The sets

$$f(x) \cup (x, y) = f(A) \cup (x, y)$$