Chapter 3. Connectedness and Compactness
Section 24. Connected Subspaces of the Real Line—Proofs of Theorems
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Theorem 24.1. If $L$ is a linear continuum in the order topology, then $L$ is connected and so are intervals and rays in $L$.

Proof. Recall that a subspace $Y$ of $L$ is convex if for every pair of points $a, b \in Y$ with $a < b$, then entire interval $[a, b] = \{x \in L \mid a \leq x \leq b\}$ lies in $Y$. 
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Let $Y$ be convex. ASSUME that $Y$ has a separation and that $Y$ is the union of disjoint nonempty sets $A$ and $B$, each of which is open in $Y$. Choose $a \in A$ and $b \in B$. WLOG, say $a < b$. Since $Y$ is convex then $[a, b] \subset Y$. 
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Theorem 24.1 (continued 1)

Proof (continued). Let $c = \sup A_0$. We now show in to cases that $c \notin A_0$ and $c \notin B_0$, which CONTRADICTS the fact that $[a, b] = A_0 \cup B_0$ (since $A_0 \subset [a, b]$ then $b$ is an upper bound for $A_0$ and so $a \leq c \leq b$ and so $c \in [a, b] = A_0 \cup B_0$). From this contradiction, it follows that $Y$ is connected.

Case 1. Suppose $c \in B_0$. Then $c \neq a$ (since $a \in A$ and $A \cap B = \emptyset$). So either $c = b$ or $a < c < b$. In either case, since $B_0$ is open in $[a, b]$ then there is some interval of the form $(d, c] \subset B_0$. 
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Theorem 24.1. If $L$ is a linear continuum in the order topology, then $L$ is connected and so are intervals and rays in $L$.

Proof (continued).

Case 2. Suppose $c \in A_0$. Then $c \neq b$ since $b \in B$. So either $c = a$ or $a < c < b$. Because $A_0$ is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in $A_0$. By property (2) of the linear continuum $L$, there is $z \in L$ such that $c < z < e$. Then $z \in A_0$, CONTRADICTING the fact that $c$ is an upper bound of $A_0$. We conclude that $c \not\in A_0$. 
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We have shown that if $Y$ is a convex subset of $L$ then $Y$ is connected. Notice that intervals and rays are convex sets and so are connected. ☐
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We have shown that if \( Y \) is a convex subset of \( L \) then \( Y \) is connected. Notice that intervals and rays are convex sets and so are connected.
Theorem 24.3. Intermediate Value Theorem.

Let \( f : X \to Y \) be a continuum map, where \( X \) is a connected space and \( Y \) is an ordered set in the order topology. If \( a \) and \( b \) are two points of \( X \) and if \( r \) is a point of \( Y \) lying between \( f(a) \) and \( f(b) \), then there exists a point \( x \in X \) such that \( f(c) = r \).

Proof. Suppose \( f, X, \) and \( Y \) are as hypothesized. The sets 
\[ A = f(X) \cap (-\infty, r) \] 
and 
\[ B = f(X) \cap (r, +\infty) \] 
are disjoint (since \((-\infty, r) \) and \((r, +\infty) \) are disjoint) and nonempty since \( f(a) \) is in one of these sets and \( f(b) \) is in the other.
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Let $f : X \to Y$ be a continuum map, where $X$ is a connected space and $Y$ is an ordered set in the order topology. If $a$ and $b$ are two points of $X$ and if $r$ is a point of $Y$ lying between $f(a)$ and $f(b)$, then there exists a point $x \in X$ such that $f(x) = r$.

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Lemma 24.A. If space $X$ is path connected then it is connected.

Proof. Let $X$ be path connected. ASSUME $X$ is not connected and that $A$ and $B$ form a separation of $X$. Let $f: [a, b] \rightarrow X$ be any path in $X$. Since $f$ is continuous and $[a, b]$ is a connected set in $\mathbb{R}$, so by Theorem 23.5, $f([a, b])$ is connected in $X$. So by Lemma 23.2, $f([a, b])$ lies either entirely in $A$ or entirely in $B$. But this cannot be the case if $a$ is chosen from $A$ and $b$ is chosen from $B$, a CONTRADICTION. So the assumption that a separation of $X$ exists is false and so space $X$ is connected.
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