Chapter 3. Connectedness and Compactness
Section 25. Components and Local Connectedness—Proofs of Theorems
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Theorem 25.1. The components of $X$ are connected disjoint subspaces of $X$ whose union is $X$, such that each nonempty connected subspace of $X$ intersects only one of them.

Proof. Since the components are by definition equivalence classes, then the components are disjoint and union to give $X$ (equivalence classes on a set partition the set; see page 23). ASSUME connected subspace $A$ of $X$ intersects two disjoint nonempty components $C_1$ and $C_2$, say at $x_1$ and $x_2$, respectively.
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Proof (continued). To show that a component $C$ is connected, let $x_0 \in C$. Then for each $x \in C$ we have $x_0 \sim x$, so there is a connected subspace $A_x$ containing $x_0$ and $x$. From the previous paragraph, a connected subspace cannot intersect two components and so $A_x \subset C$. Therefore, $C = \bigcup_{x \in C} A_x$. Since each $A_x$ is connected and $x_0 \in A_x$ for all $x \in C$ then by Theorem 23.3, $C$ is connected.
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Lemma 25.A. Each connected component of a space $X$ is closed. If $X$ has only finitely many connected components, then each component of $X$ is also open.

Proof. Let $C$ be a connected component of $X$. By Theorem 23.4, $\overline{C}$ is also connected. Since the components are disjoint by Theorem 25.1, then $C = \overline{C}$ and so $C$ is closed by Lemma 17.A.
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If $X$ has only finitely many components then the complement of a component $C$ is a finite union of closed sets by the first part of this lemma, and so the complement of $C$ is closed by Theorem 17.1. Hence $C$ is open.
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Theorem 25.3. A space $X$ is locally connected if and only if for every open set $U$ of $X$, each component of $U$ is open in $X$.

Proof. Suppose $X$ is locally connected. Let $U$ be an open set in $X$, let $C$ be a connected component of $U$, and let $x \in C$. Then by the definition of locally connected, there is a connected neighborhood $V$ of $x$ with $V \subset U$. Since $V$ is connected, by Theorem 25.1, it must lie entirely in the component $C$, $V \subset C$. So $C$ is open.

Conversely, suppose that the components of open sets in $X$ are open. Let $x \in X$ and let $U$ be an arbitrary neighborhood of $x$. Let $C$ be the connected component of $U$ which contains $x$. Now $C$ is connected and, by hypothesis, open in $X$. So, by definition, $X$ is locally connected.
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**Theorem 25.5.** If $X$ is a topological space, each path component of $X$ lies in a component of $X$. If $X$ is locally path connected, then the component and the path components are the same.

**Proof.** Let $C$ be a component of $X$. Let $x \in C$. Let $P$ be the path component of $X$ containing $x$. By Lemma 24.A, $P$ is connected and so $P \subset C$ by Theorem 25.1, and the first claim holds.
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Suppose $X$ is locally path connected. **ASSUME** $P \neq C$. Let $Q$ denote the union of all the path components of $X$ that are different from $P$ and which intersect $C$ (since $P \neq C$ then $Q \neq \emptyset$). As above, by Lemma 24.A and Theorem 25.1, each of these path components must be in component $C$. 
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