Theorem 29.1 (continued I)

If $X$ and $Y$ are two spaces satisfying the following conditions:

1. $X$ is a subspace of a single point.
2. The set $X \setminus \{p\}$ consists of a single point.
3. $Y$ is a compact Hausdorff space.

Then $X$ is a locally compact Hausdorff space.

Proof. We follow Munkres’ three-step proof (which, oddly enough, does not concern the numbered conditions of $\Lambda$).

Step 1. We first verify the homeomorphism claim. Let $\Lambda$ and $Y$ be spaces.

Corollary. For any $\lambda \in \Lambda$, $\Lambda \cap \{\lambda\}$ is open.

Proof follows from Theorem 29.1.

Section 29. Local Compactness and Compactness

Chapter 3. Connectedness and Compactness

Introduction to Topology
Theorem 2.9.2

Condition (3) holds.

Theorem 2.9.1 (continued 3)

Let \( A \subseteq Y \subseteq X \) and \( Y \) open in \( X \).

Theorem 2.9.1 (continued 4)

Let \( \mathcal{U} \) be a finite open cover of \( A \) by \( \mathcal{U} \) and \( Y \) is open in \( X \).

Proof (continued). Similarly, we have closure under unions.
Corollary 29.4

A space $X$ is locally compact and Hausdorff if and only if $X$ is open in compact Hausdorff space $Y$.

Since $\emptyset = Y \setminus X$, and thus is a closed set by Theorem 17.8, then $X$ is open in $Y$.

**Proof.** By Theorem 29.1, $X$ is locally compact and Hausdorff if and only if $X$ is open in compact Hausdorff space $Y$.

**Corollary 29.4.**

$X$ is locally compact if and only if $X$ is open in compact Hausdorff space $Y$.

**Proof.** Suppose $A$ is open in $X$. Let $x \in A$. Then $A = \Delta \cup C$.

**Theorem 29.2** (continued). Then $\Delta$ is compact (again, by Theorem 29.2) and $\Lambda$, which can be done by compactness.

**Corollary 29.3.** Let $X$ be locally compact and Hausdorff. Let $A$ be a compact subspace of $X$. Then $A$ is locally compact.

**Theorem 29.2** (continued).

If $A$ is compact and contains the neighborhood $V \cup A$ of $x$, then $A$ is locally compact.

**Proof.** Suppose $A$ is closed in $X$. Given $x \in A$, let $C$ be a compact subspace of $X$. Then $A$ is locally compact.

are points of closure of $A$. So $\Lambda \supseteq \Delta \cup C$ and thus is the desired set. Set $\Lambda$ and $\Delta \subseteq \Lambda \cup C$, and thus $\Lambda \setminus \Delta \subseteq \Lambda \setminus C$.

**Corollary 29.4.** A space $X$ is Hausdorff if and only if $X$ is a subspace of a compact Hausdorff space $Y$.