Chapter 3. Connectedness and Compactness
Section 29. Local Compactness—Proofs of Theorems
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**Theorem 29.1.** Let $X$ be a topological space. Then $X$ is a locally compact Hausdorff space if and only if there is a topological space $Y$ satisfying the following conditions:

1. $X$ is a subspace of $Y$.
2. The set $Y \setminus X$ consists of a single point.
3. $Y$ is a compact Hausdorff space.

If $Y$ and $Y'$ are two spaces satisfying these conditions, then there is a homeomorphism of $Y$ with $Y'$ that equals the identity map on $X$.

**Proof.** We follow Munkres’ three-step proof (which oddly enough does not correspond to the numbered conditions of $Y$).
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**Proof.** We follow Munkres’ three-step proof (which oddly enough does not correspond to the numbered conditions of $Y$).

Step 1. We first verify the homeomorphism claim. Let $Y$ and $Y'$ be spaces satisfying the three conditions. Define $Y \rightarrow Y'$ by letting $h$ map the “single point” $p \in Y \setminus X$ to the “single point” $q \in Y' \setminus X$, and letting $h$ equal the identity on $X$. Then $h$ is a bijection (one to one and onto).
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Proof (continued). Let $U$ be an open set in $Y$. First, suppose $p \notin U$. Then $h(U) = U$ is open in $X$ (under the subspace topology). Now $X$ is open in $Y'$ since $Y \setminus X$ is closed ($Y' \setminus X$ is a singleton, which forms a closed set, by Theorem 17.8, because $Y$ is compact by (3)) and so $U$ is open in $Y'$. Second, suppose $p \in U$. Since $C = Y \setminus U$ is closed in $Y$, then $C$ is a compact subspace of $Y$, by Theorem 26.2, since $Y$ is compact by (3). Since $C \subset X$, $C$ is also compact in $X$. Since $X \subset Y'$, the space $C$ is also a compact subspace of $Y'$ (every open covering of $C$ with sets open in $Y'$ yields an open covering of $C$ with sets open in $X$ under the subspace topology—and hence finite subcovers). Since $Y'$ is Hausdorff by (3), Theorem 26.3 implies that $C$ is closed in $Y'$, and so $Y' \setminus C$ is open. But $h(U) = U \cup \{q\} = (Y \setminus C) \cup \{q\} = Y' \setminus C$ and so $h(U)$ is open. In both cases, for any open $U$ we have that $h(U)$ is open and so $h^{-1}$ is continuous. Interchanging $Y$ and $Y'$ shows that $h$ is continuous and therefore $h$ is a homeomorphism.
Theorem 29.1 (continued 1)

**Proof (continued).** Let $U$ be an open set in $Y$. First, suppose $p \notin U$. Then $h(U) = U$ is open in $X$ (under the subspace topology). Now $X$ is open in $Y'$ since $Y \setminus X$ is closed ($Y' \setminus X$ is a singleton, which forms a closed set, by Theorem 17.8, because $Y$ is compact by (3)) and so $U$ is open in $Y'$. Second, suppose $p \in U$. Since $C = Y \setminus U$ is closed in $Y$, then $C$ is a compact subspace of $Y$, by Theorem 26.2, since $Y$ is compact by (3). Since $C \subset X$, $C$ is also compact in $X$. Since $X \subset Y'$, the space $C$ is also a compact subspace of $Y'$ (every open covering of $C$ with sets open in $Y'$ yields an open covering of $C$ with sets open in $X$ under the subspace topology—and hence finite subcovers).
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Proof (continued).

Step 2. Suppose $X$ is locally compact and Hausdorff. We construct set $Y$ by adding a single element to $X$, say $Y = X \cup \{\infty\}$. This gives condition (2). Define the collection of subsets of $Y$, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ where $\mathcal{T}_1 = \{U \subset X \mid U \text{ is open in } X\}$ and $\mathcal{T}_2 = \{Y \setminus C \mid C \subset X \text{ is compact in } X\}$. 
Proof (continued).
Step 2. Suppose $X$ is locally compact and Hausdorff. We construct set $Y$ by adding a single element to $X$, say $Y = X \cup \{\infty\}$. This gives condition (2). Define the collection of subsets of $Y$, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ where $\mathcal{T}_1 = \{U \subset X \mid U \text{ is open in } X\}$ and $\mathcal{T}_2 = \{Y \setminus C \mid C \subset X \text{ is compact in } X\}$. We now show that $\mathcal{T}$ is a topology on $Y$. Since $\emptyset$ is open and compact in $X$, then $\emptyset, Y \in \mathcal{T}$. For closure of $\mathcal{T}$ under intersections we consider three cases:

\begin{align*}
U_1 \cap U_2 & \in \mathcal{T}_1 \\
(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) & \in \mathcal{T}_2 \\
U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) & \in \mathcal{T}_1.
\end{align*}
Theorem 29.1 (continued 2)

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Step 2. Suppose $X$ is locally compact and Hausdorff. We construct set $Y$ by adding a single element to $X$, say $Y = X \cup \{\infty\}$. This gives condition (2). Define the collection of subsets of $Y$, $\mathcal{T} = T_1 \cup T_2$ where

$T_1 = \{U \subset X \mid U \text{ is open in } X\}$ and

$T_2 = \{Y \setminus C \mid C \subset X \text{ is compact in } X\}$. We now show that $\mathcal{T}$ is a topology on $Y$. Since $\emptyset$ is open and compact in $X$, then $\emptyset, Y \in \mathcal{T}$. For closure of $\mathcal{T}$ under intersections we consider three cases:

\[
U_1 \cap U_2 \in T_1
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(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) \in T_2
\]

\[
U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) \in T_1.
\]
Theorem 29.1 (continued 3)

Proof (continued). Similarly, we have closure under unions:

\[ \bigcup U_\alpha = U \in T_1 \]
\[ \bigcup (Y \setminus C_\beta) = Y \setminus (\cap C_\beta) = V \setminus C \in T_2 \]
\[ (\bigcup U_\alpha) \cup (\bigcup Y \setminus C_\beta) = U \cup (Y \setminus C) = T \setminus (C \setminus U) \in T_2. \]

Now we show that \( X \) is a subspace of \( Y \) (confirming condition (1)). Given any open set \( U \) of \( Y \), we need to show that \( X \cap U \) is open in \( X \). If \( U \in T_1 \) then \( U \cap X = U \); if \( U = Y \setminus C \in T_2 \) then \( (Y \setminus C) \cap X = X \setminus C \in T_2 \).
Theorem 29.1 (continued 3)

Proof (continued). Similarly, we have closure under unions:

\[
\bigcup U_\alpha = U \in T_1
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\bigcup (Y \setminus C_\beta) = Y \setminus \bigcap C_\beta = V \setminus C \in T_2
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Now we show that $X$ is a subspace of $Y$ (confirming condition (1)). Given any open set $U$ of $Y$, we need to show that $X \cap U$ is open in $X$. If $U \in T_1$ then $U \cap X = U$; if $U = Y \setminus C \in T_2$ then $(Y \setminus C) \cap X = X \setminus C \in T_2$.

Conversely, any open set in $X$ is in $T_1$ and therefore is open in $Y$. So the topology on $X$ is the same as the subspace topology on $X$ as a subspace of $Y$. That is, $X$ is a subspace of $Y$ and condition (1) holds.
Proof (continued). Similarly, we have closure under unions:

\[ \bigcup U_\alpha = U \in T_1 \]

\[ \bigcup (Y \setminus C_\beta) = Y \setminus (\bigcap C_\beta) = V \setminus C \in T_2 \]

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Conversely, any open set in \( X \) is in \( T_1 \) and therefore is open in \( Y \). So the topology on \( X \) is the same as the subspace topology on \( X \) as a subspace of \( Y \). That is, \( X \) is a subspace of \( Y \) and condition (1) holds.
Theorem 29.1 (continued 4)

Proof (continued). Now we show that \( Y \) is compact. Let \( \mathcal{A} \) be an open covering of \( Y \). Since \( \infty \) must be in some element of \( \mathcal{A} \), then there is compact \( C \subset X \) such that \( Y \setminus C \in T_2 \) is in \( \mathcal{A} \). Since \( C \) is compact and \( \mathcal{A} \) is a covering of \( C \) then there is a finite subcover \( \mathcal{A}' \) of \( \mathcal{A} \) which covers \( C \).
Theorem 29.1 (continued 4)

Proof (continued). Now we show that $Y$ is compact. Let $A$ be an open covering of $Y$. Since $\infty$ must be in some element of $A$, then there is compact $C \subset X$ such that $Y \setminus C \in T_2$ is in $A$. Since $C$ is compact and $A$ is a covering of $C$ then there is a finite subcover $A'$ of $A$ which covers $C$. Then $A' \cup \{Y \setminus C\}$ is a finite cover of $C$. Then $A' \cup \{Y \setminus C\}$ is a finite cover of $Y$. Hence $Y$ is compact.
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Next, we show that $Y$ is Hausdorff. Let $x, y \in Y$ with $x \neq y$. 
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Next, we show that \( Y \) is Hausdorff. Let \( x, y \in Y \) with \( x \neq y \). If \( x \) and \( y \) are both in \( X \), then there are disjoint open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \), respectively, since \( X \) is Hausdorff. If \( x \in X \) and \( y = \infty \) then, since \( X \) is hypothesized to be locally compact, there is compact \( C \) in \( X \) containing neighborhood \( U \) of \( x \).
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Proof (continued). Now we show that $Y$ is compact. Let $\mathcal{A}$ be an open covering of $Y$. Since $\infty$ must be in some element of $\mathcal{A}$, then there is compact $C \subset X$ such that $Y \setminus C \in T_2$ is in $\mathcal{A}$. Since $C$ is compact and $\mathcal{A}$ is a covering of $C$ then there is a finite subcover $\mathcal{A}'$ of $\mathcal{A}$ which covers $C$. Then $\mathcal{A}' \cup \{Y \setminus C\}$ is a finite cover of $C$. Then $\mathcal{A}' \cup \{Y \setminus C\}$ is a finite cover of $Y$. Hence $Y$ is compact.

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Theorem 29.1 (continued 5)

Proof (continued).
Step 3. We now show the converse. Suppose $Y$ satisfies conditions (1), (2), and (3). Then $X$ is Hausdorff because it is a subspace of $Y$ (and it has the subspace topology). Let $x \in X$ be given. Since $Y$ is Hausdorff, there are disjoint open sets $U$ and $V$ in $Y$ containing $\infty$ and the single point of $Y \setminus X = \{\infty\}$, respectively.
Proof (continued).

Step 3. We now show the converse. Suppose $Y$ satisfies conditions (1), (2), and (3). Then $X$ is Hausdorff because it is a subspace of $Y$ (and it has the subspace topology). Let $x \in X$ be given. Since $Y$ is Hausdorff, there are disjoint open sets $U$ and $V$ in $Y$ containing $\infty$ and the single point of $Y \setminus X = \{\infty\}$, respectively. The set $C = Y \setminus V$ is closed in $Y$ and so is compact since $Y$ is compact (by Theorem 26.2). Since $\infty \in V$ then $\infty \notin C = Y \setminus V$ and so $C \subset X$ is also compact in $X$ (since $X$ has the subspace topology by (1)). Also, $C$ contains neighborhood $U$ of $x$, and so $X$ is locally compact.
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Step 3. We now show the converse. Suppose $Y$ satisfies conditions (1), (2), and (3). Then $X$ is Hausdorff because it is a subspace of $Y$ (and it has the subspace topology). Let $x \in X$ be given. Since $Y$ is Hausdorff, there are disjoint open sets $U$ and $V$ in $Y$ containing $\infty$ and the single point of $Y \setminus X = \{\infty\}$, respectively. The set $C = Y \setminus V$ is closed in $Y$ and so is compact since $Y$ is compact (by Theorem 26.2). Since $\infty \in V$ then $\infty \notin C = Y \setminus V$ and so $C \subset X$ is also compact in $X$ (since $X$ has the subspace topology by (1). Also, $C$ contains neighborhood $U$ of $x$, and so $X$ is locally compact. □
Theorem 29.2

**Theorem 29.2.** Let $X$ be a Hausdorff space. Then $X$ is locally compact if and only if given $x \in X$, and given a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subset U$.

**Proof.** If $X$ satisfies this condition, then certainly there is a compact subspace of $X$ (namely $\overline{V}$) containing a neighborhood $V$ of $x$; that is, the condition implies locally compact.
Theorem 29.2. Let $X$ be a Hausdorff space. Then $X$ is locally compact if and only if given $x \in X$, and given a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof. If $X$ satisfies this condition, then certainly there is a compact subspace of $X$ (namely $\overline{V}$) containing a neighborhood $V$ of $x$; that is, the condition implies locally compact.

Conversely, suppose $X$ is locally compact and let $x \in X$ with $U$ a neighborhood of $x$. Since $S$ is locally compact, by Theorem 29.1 there is a space $Y$, the one-point compactification of $X$. Let $C = Y \setminus U$. Since $U$ is open in $X$ then $U$ is open in $Y$ (in the proof of Theorem 29.1, all sets open in $X$ are open in $Y$) and so $C$ is closed in $Y$. 
Theorem 29.2. Let $X$ be a Hausdorff space. Then $X$ is locally compact if and only if given $x \in X$, and given a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subset U$.

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Theorem 29.2

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Theorem 29.2. Let $X$ be a Hausdorff space. Then $X$ is locally compact if and only if given $x \in X$, and given a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof (continued). Then $\overline{V}$ is compact (again, by Theorem 26.2) and $\overline{V}$ is disjoint from $C$ since $\overline{V} = V \cup V'$ where $V'$ is the set of limit point of set $V$, and since $x \in V$, $C \subset W$, and $V \cap W = \emptyset$, then no points of $C$ are points of closure of $V$. So $\overline{V} \subset T \setminus C = U$ is the desired set.
Corollary 29.3.

Let $X$ be locally compact and Hausdorff. Let $A$ be a subspace of $X$. If $A$ is closed in $X$ or open in $X$, then $A$ is locally compact.

**Proof.** Suppose $A$ is closed in $X$. Given $x \in A$, let $C$ be a compact subspace of $X$ containing neighborhood $U$ of $x \in X$ (which can be done since $X$ is locally compact).
Corollary 29.3. Let $X$ be locally compact and Hausdorff. Let $A$ be a subspace of $X$. If $A$ is closed in $X$ or open in $X$, then $A$ is locally compact.

**Proof.** Suppose $A$ is closed in $X$. Given $x \in A$, let $C$ be a compact subspace of $X$ containing neighborhood $U$ of $x \in X$ (which can be done since $X$ is locally compact). Then $C \cap A$ is closed in $C$ and thus (by Theorem 26.2) compact and it contains the neighborhood $U \cap A$ of $x \in A$. That is, $A$ is locally compact.
Corollary 29.3

**Corollary 29.3.** Let $X$ be locally compact and Hausdorff. Let $A$ be a subspace of $X$. If $A$ is closed in $X$ or open in $X$, then $A$ is locally compact.

**Proof.** Suppose $A$ is closed in $X$. Given $x \in A$, let $C$ be a compact subspace of $X$ containing neighborhood $U$ of $x \in X$ (which can be done since $X$ is locally compact). Then $C \cap A$ is closed in $C$ and thus (by Theorem 26.2) compact and it contains the neighborhood $U \cap A$ of $x \in A$. That is, $A$ is locally compact.

Suppose $A$ is open in $X$. Let $x \in A$. By Theorem 29.2, there is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subset A$. 
Corollary 29.3

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Corollary 29.3. Let $X$ be locally compact and Hausdorff. Let $A$ be a subspace of $X$. If $A$ is closed in $X$ or open in $X$, then $A$ is locally compact.

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Corollary 29.4 A space $X$ is homeomorphic to an open subspace of a compact Hausdorff space if and only if $X$ is locally compact and Hausdorff.

Proof. By Theorem 29.1, $X$ is locally compact and Hausdorff if and only if it has a one-point compactification $Y$, which is compact and Hausdorff.
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