Theorem 3.2.1. Every regular space with a countable basis is normal.

Proof. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is regular, there is a basis element $A'$ of $A$ containing $x$, for which $x \notin B$. Similarly, there is a basis element $B'$ of $B$ containing $y$, for which $y \notin A$. Then $A' \cap B'$ is empty.

Let $d(A')$ and $d(B')$ be disjoint open sets with $A' \subset d(A')$ and $B' \subset d(B')$. Then $X \setminus (d(A') \cup d(B'))$ is open and $d(A') \cup d(B') \supseteq X \setminus (d(A') \cup d(B'))$, so $X \setminus (d(A') \cup d(B')) = X \setminus d(A') \cap X \setminus d(B')$ is open.

Now, $A' \cup B'$ is closed, so $X \setminus (A' \cup B')$ is open. Since $A' \cup B' \subset X \setminus (d(A') \cup d(B'))$, we have $A \subset X \setminus (A' \cup B')$ and $B \subset X \setminus (d(A') \cup d(B'))$. This shows that $A$ and $B$ are separated.

Theorem 3.2.2. Every metrizable space is normal.

Proof. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is normal, there is an open set $U$ containing $A$ and an open set $V$ containing $B$ such that $U \cap V = \emptyset$. Then $U$ and $V$ are disjoint open sets with $A \subset U$ and $B \subset V$. Setting $W = X \setminus (U \cup V)$, we have $W$ open and $A \subset W$ and $B \subset W$. Thus $A$ and $B$ are separated.

Section 3.2. Normal Spaces—Proofs of Theorems

Chapter 4. Compactness and Separation Axioms

Introduction to Topology
Theorem 3.2.4 (continued)

Theorem 3.2.4


Proof (continued). Every well-ordered set $X$ is normal in the order topology.

}\hfill\Box

\[ B \subseteq \bigcup_{\alpha \in \beta} A_{\alpha} \subseteq \bigcup_{\alpha \in \beta} V_{\alpha} = U \]

\[ \text{exists (a) disjoint open sets with } A \cap B = \emptyset. \]

\[ \text{Therefore, } X \text{ is normal.} \]

\[ \text{Similary, if } \epsilon < \delta \text{ then } \epsilon \notin \delta(\epsilon) \text{ and } \epsilon \notin \delta(\epsilon). \]

\[ \text{Similarily, if } \epsilon < \delta \text{ then } \epsilon \notin \delta(\epsilon) \text{ and } \epsilon \notin \delta(\epsilon). \]

\[ \text{Therefore, } X \text{ is normal.} \]

\[ \text{Similary, if } \epsilon < \delta \text{ then } \epsilon \notin \delta(\epsilon) \text{ and } \epsilon \notin \delta(\epsilon). \]
Hence, $X$ is normal. Condition is satisfied when one of $A$ or $B$ contains the smallest element of $X$.

Disjoint open sets containing $A$ and $B$ respectively. So the normality condition is satisfied when $A \cap B = \emptyset$ and $A \cup B = X$. The smallest element of $A$ is contained in $X$. The smallest element of $A$ is contained in $X$. The smallest element of $A$ is contained in $X$. The smallest element of $A$ is contained in $X$.

Finally, suppose $A$ and $B$ are disjoint closed sets in $X$ where $A$ contains the smallest element of $X$ and $B$ contains the smallest element of $X$.

**Proof (continued).** So the normality condition is satisfied when neither (closed) nor $B$ contains the smallest element of $X$.

**Theorem 3.2.4.** Every well-ordered set $X$ is normal in the order topology.