Introduction to Topology

Chapter 4. Countability and Separation Axioms
Section 32. Normal Spaces—Proofs of Theorems
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Theorem 32.1.

Every regular space with a countable basis is normal.

Proof. Let $X$ be a regular space with a countable basis $B$. Let $A$ and $B$ be disjoint closed sets in $X$. 

Since $X$ is regular, each $x \in A$ has a neighborhood $U$ not intersecting $B$. By Lemma 31.1(a), there is a neighborhood $V$ of $x$ with $V \subset U$, and there is a basis element of $B$ containing $x$ which is a subset of $V$. Choose such a basis element for each $x \in A$. Then this is a countable (since $B$ is countable) covering of $A$ by open sets whose closures do not intersect $B$. Denote the sets in this covering as $\{U_n\}_{n \in \mathbb{N}}$.

Similarly, find a countable collection $\{V_n\}$ of open sets covering $B$ such that each set $V_n$ is disjoint from $A$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ are open sets containing $A$ and $B$, respectively (but they may not be disjoint).

Now, for $n \in \mathbb{N}$, define $U'_n = U_n \cup \bigcup_{i=1}^{n} V_i$ and $V'_n = V_n \cup \bigcup_{i=1}^{n} U_i$. Then each $U'_n$ and $V'_n$ is open.
Theorem 32.1

**Theorem 32.1.** Every regular space with a countable basis is normal.

**Proof.** Let $X$ be a regular space with a countable basis $\mathcal{B}$. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is regular, each $x \in A$ has a neighborhood $U$ not intersecting $B$. By Lemma 31.1(a), there is a neighborhood $V$ of $x$ with $\overline{V} \subseteq U$, and there is a basis element of $\mathcal{B}$ containing $x$ which is a subset of $V$. Then this is a countable (since $\mathcal{B}$ is countable) covering of $A$ by open sets whose closures do not intersect $B$. Denote the sets in this covering as $\{U_n\}_{n \in \mathbb{N}}$. Similarly, find a countable collection $\{V_n\}$ of open sets covering $B$ such that each set $V_n$ is disjoint from $A$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ are open sets containing $A$ and $B$, respectively (but they may not be disjoint). Now, for $n \in \mathbb{N}$, define $U'_n = U_n \cup \bigcup_{i=1}^{n} V_i$ and $V'_n = V_n \cup \bigcup_{i=1}^{n} U_i$. Then each $U'_n$ and $V'_n$ is open.
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**Proof.** Let $X$ be a regular space with a countable basis $\mathcal{B}$. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is regular, each $x \in A$ has a neighborhood $U$ not intersecting $B$. By Lemma 31.1(a), there is a neighborhood $V$ of $x$ with $\overline{V} \subset U$, and there is a basis element of $\mathcal{B}$ containing $x$ which is a subset of $V$. Choose such a basis element for each $x \in A$. Then this is a countable (since $\mathcal{B}$ is countable) covering of $A$ by open sets whose closures do not intersect $B$. Denote the sets in this covering as $\{U_n\}_{n \in \mathbb{N}}$. 
Theorem 32.1

Theorem 32.1. Every regular space with a countable basis is normal.

Proof. Let $X$ be a regular space with a countable basis $B$. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is regular, each $x \in A$ has a neighborhood $U$ not intersecting $B$. By Lemma 31.1(a), there is a neighborhood $V$ of $x$ with $\overline{V} \subset U$, and there is a basis element of $B$ containing $x$ which is a subset of $V$. Choose such a basis element for each $x \in A$. Then this is a countable (since $B$ is countable) covering of $A$ by open sets whose closures do not intersect $B$. Denote the sets in this covering as $\{U_n\}_{n \in \mathbb{N}}$.

Similarly, find a countable collection $\{V_n\}$ of open sets covering $B$ such that each set $\overline{V}_n$ is disjoint from $A$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ are open sets containing $A$ and $B$, respectively (but they may not be disjoint).
Theorem 32.1

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Proof. Let $X$ be a regular space with a countable basis $\mathcal{B}$. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is regular, each $x \in A$ has a neighborhood $U$ not intersecting $B$. By Lemma 31.1(a), there is a neighborhood $V$ of $x$ with $\overline{V} \subset U$, and there is a basis element of $\mathcal{B}$ containing $x$ which is a subset of $V$. Choose such a basis element for each $x \in A$. Then this is a countable (since $\mathcal{B}$ is countable) covering of $A$ by open sets whose closures do not intersect $B$. Denote the sets in this covering as $\{U_n\}_{n \in \mathbb{N}}$. Similarly, find a countable collection $\{V_n\}$ of open sets covering $B$ such that each set $\overline{V}_n$ is disjoint from $A$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ are open sets containing $A$ and $B$, respectively (but they may not be disjoint). Now, for $n \in \mathbb{N}$, define

$$U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V}_i \quad \text{and} \quad V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U}_i.$$ 

Then each $U'_n$ and $V'_n$ is open.
Theorem 32.1

Theorem 32.1. Every regular space with a countable basis is normal.

Proof. Let $X$ be a regular space with a countable basis $B$. Let $A$ and $B$ be disjoint closed sets in $X$. Since $X$ is regular, each $x \in A$ has a neighborhood $U$ not intersecting $B$. By Lemma 31.1(a), there is a neighborhood $V$ of $x$ with $\overline{V} \subset U$, and there is a basis element of $B$ containing $x$ which is a subset of $V$. Choose such a basis element for each $x \in A$. Then this is a countable (since $B$ is countable) covering of $A$ by open sets whose closures do not intersect $B$. Denote the sets in this covering as $\{U_n\}_{n \in \mathbb{N}}$.

Similarly, find a countable collection $\{V_n\}$ of open sets covering $B$ such that each set $\overline{V}_n$ is disjoint from $A$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ are open sets containing $A$ and $B$, respectively (but they may not be disjoint). Now, for $n \in \mathbb{N}$, define

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i.$$ 

Then each $U'_n$ and $V'_n$ is open.
Theorem 32.1. Every regular space with a countable basis is normal.

Proof (continued). The collection $\{U'_n\}$ covers $A$ and $\{V'_n\}$ covers $B$ (this is where the "$\overline{U}_n$ is disjoint from $B$" and "$\overline{V}_n$ is disjoint from $A$" parts are used).

Finally, consider $U' = \bigcup_{n \in \mathbb{N}} U'_n$ and $V' = \bigcup_{n \in \mathbb{N}} V'_n$. Assume $x \in U' \cap V'$. Then $x \in U'_j \cap V'_k$ for some $j, k \in \mathbb{N}$. 
Theorem 32.1. Every regular space with a countable basis is normal.

Proof (continued). The collection \( \{ U'_n \} \) covers \( A \) and \( \{ V'_n \} \) covers \( B \) (this is where the “\( \overline{U}_n \) is disjoint from \( B \)” and “\( \overline{V}_n \) is disjoint from \( A \)” parts are used).

Finally, consider \( U' = \bigcup_{n \in \mathbb{N}} U'_n \) and \( V' = \bigcup_{n \in \mathbb{N}} V'_n \). Assume \( x \in U' \cap V' \). Then \( x \in U'_j \cap V'_k \) for some \( j, k \in \mathbb{N} \). If \( j \leq k \) then \( x \in U_j \) (since \( U'_j = J_j \setminus \bigcup_{i=1}^{j} \overline{V}_i \)) but, since \( j \leq k \), \( x \notin V_k \) (since \( V'_k = V_k \setminus \bigcup_{i=1}^{k} \overline{U}_i \), a CONTRADICTION.
Theorem 32.1 (continued)

**Theorem 32.1.** Every regular space with a countable basis is normal.

**Proof (continued).** The collection \( \{U'_n\} \) covers \( A \) and \( \{V'_n\} \) covers \( B \) (this is where the “\( \overline{U}_n \) is disjoint from \( B \)” and “\( \overline{V}_n \) is disjoint from \( A \)” parts are used).

Finally, consider \( U' = \bigcup_{n \in \mathbb{N}} U'_n \) and \( V' = \bigcup_{n \in \mathbb{N}} V'_n \). **ASSUME** \( x \in U' \cap V' \). Then \( x \in U'_j \cap V'_k \) for some \( j, k \in \mathbb{N} \). If \( j \leq k \) then \( x \in U_j \) (since \( U'_j = J_j \setminus \bigcup_{i=1}^{j} \overline{V}_i \)) but, since \( j \leq k \), \( x \notin V_k \) (since \( V'_k = V_k \setminus \bigcup_{i=1}^{k} \overline{U}_i \)), a **CONTRADICTION.** A similar contradiction follows if \( j \geq k \). So \( U' \) and \( V' \) are disjoint open sets with \( A \subset U' \) and \( B \subset V' \). That is, \( X \) is regular. \( \square \)
Theorem 32.1. Every regular space with a countable basis is normal.

Proof (continued). The collection \( \{ U'_n \} \) covers \( A \) and \( \{ V'_n \} \) covers \( B \) (this is where the “\( \overline{U}_n \) is disjoint from \( B \)” and “\( \overline{V}_n \) is disjoint from \( A \)” parts are used).

Finally, consider \( U' = \bigcup_{n \in \mathbb{N}} U'_n \) and \( V' = \bigcup_{n \in \mathbb{N}} V'_n \). Assume \( x \in U' \cap V' \). Then \( x \in U'_j \cap V'_k \) for some \( j, k \in \mathbb{N} \). If \( j \leq k \) then \( x \in U_j \) (since \( U'_j = J_j \setminus \bigcup_{i=1}^{j} \overline{V}_i \)) but, since \( j \leq k \), \( x \not\in V_k \) (since \( V'_k = V_k \setminus \bigcup_{i=1}^{k} \overline{U}_i \)), a contradiction. A similar contradiction follows if \( j \geq k \). So \( U' \) and \( V' \) are disjoint open sets with \( A \subset U' \) and \( B \subset V' \). That is, \( X \) is regular. \( \square \)
Theorem 32.2

Theorem 32.2. Every metrizable space is normal.

Proof. Let $X$ be metrizable with metric $d$. Let $A$ and $B$ be disjoint closed sets in $X$. For each $a \in A$, choose $\varepsilon_a > 0$ so that $B(a, \varepsilon_a)$ does not intersect $B$. Similarly, for each $b \in B$, choose $\varepsilon_b > 0$ so that $B(b, \varepsilon_b)$ does not intersect $A$. Define $U = \bigcup_{a \in A} B(a, \varepsilon_a/2)$ and $V = \bigcup_{b \in B} B(b, \varepsilon_b/2)$. Then $U$ and $V$ are open sets and $A \subset U$, $B \subset V$. Assume $z \in U \cap V$. Then $z \in B(a, \varepsilon_a/2)$ and $z \in B(b, \varepsilon_b/2)$ for some $a \in A$ and $b \in B$. By the Triangle Inequality, $d(a, b) \leq d(a, z) + d(z, b) < \varepsilon_a/2 + \varepsilon_b/2$. If $\varepsilon_a \leq \varepsilon_b$ then $d(a, b) < \varepsilon_b$ and then $a \in B(b, \varepsilon_b)$, a contradiction.
**Theorem 32.2**

**Theorem 32.2.** Every metrizable space is normal.

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$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \text{ and } V = \bigcup_{b \in B} B(b, \varepsilon_b/2).$$

Then $U$ and $V$ are open sets and $A \subset U$, $B \subset V$. 
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$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \text{ and } V = \bigcup_{b \in B} B(b, \varepsilon_b/2).$$

Then $U$ and $V$ are open sets and $A \subset U$, $B \subset V$. **ASSUME** $z \in U \cap V$. Then $z \in B(a, \varepsilon_a/2)$ and $z \in B(b, \varepsilon_b/2)$ for some $a \in A$ and $b \in B$. By the Triangle Inequality,

$$d(a, b) \leq d(a, z) + d(z, b) < \varepsilon_a/2 + \varepsilon_b/2.$$

If $\varepsilon_a \leq \varepsilon_b$ then $d(a, b) < \varepsilon_b$ and then $a \in B(b, \varepsilon_b)$, a **CONTRADICTION**.
Theorem 32.2

**Theorem 32.2.** Every metrizable space is normal.

**Proof.** Let $X$ be metrizable with metric $d$. Let $A$ and $B$ be disjoint closed sets in $X$. For each $a \in A$, choose $\varepsilon_a > 0$ so that $B(a, \varepsilon_a)$ does not intersect $B$ (since $B$ is closed, it contains its limit points by Corollary 17.7, so $a$ is not a limit point of $B$ and such $B(a, \varepsilon_a)$ exists). Similarly, for each $b \in B$ choose $\varepsilon_b > 0$ so that $B(b, \varepsilon_b)$ does not intersect $A$. Define

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \text{ and } V = \bigcup_{b \in B} B(b, \varepsilon_b/2).$$

Then $U$ and $V$ are open sets and $A \subset U$, $B \subset V$. **ASSUME** $z \in U \cap V$. Then $z \in B(a, \varepsilon_a/2)$ and $z \in B(b, \varepsilon_b/2)$ for some $a \in A$ and $b \in B$. By the Triangle Inequality,

$$d(a, b) \leq d(a, z) + d(z, b) < \varepsilon_a/2 + \varepsilon_b/2.$$

If $\varepsilon_a \leq \varepsilon_b$ then $d(a, b) < \varepsilon_b$ and then $a \in B(b, \varepsilon_b)$, a **CONTRADICTION.**
Theorem 32.2. Every metrizable space is normal.

Proof (continued). Similarly, if $\varepsilon_b \leq \varepsilon_a$ then $d(a, b) < \varepsilon_a$ and $b \in B(a, \varepsilon_a)$, a contradiction. So the assumption that such $z \in U \cap V$ exists is false and $U$ and $V$ are disjoint open sets with $A \subset U$ and $B \subset V$. Therefore, $X$ is normal.
Theorem 32.3. Every compact Hausdorff space is normal.

Proof. Let $X$ be a compact Hausdorff space. Let $A$ and $B$ be disjoint closed sets in $X$. Let $x \in A$ and $y \in B$. Since $A$ is closed and $X$ is Hausdorff, by Theorem 26.2, $A$ is compact. Let $\{U_a\}_{a \in A}$ be a disjoint open covering of $A$. Since $A$ is compact, there exists a finite subcovering $\{U_1, U_2, \ldots, U_n\}$ of $A$. Let $U = U_1 \cap U_2 \cap \cdots \cap U_n$ and $V = V_1 \cap V_2 \cap \cdots \cap V_n$ be disjoint open sets where $A \subset U$ and $B \subset V$. That is, $X$ is regular.
Theorem 32.3. Every compact Hausdorff space is normal.

Proof. Let $X$ be a compact Hausdorff space. Let $A$ and $B$ be disjoint closed sets in $X$. By Lemma 26.4, for each $a \in A$, there are disjoint open $U_a$ and $V_a$ with $x \in U_x$ and $B \subset V_x$. Since $A$ is closed and $X$ is Hausdorff, then $A$ is compact by Theorem 26.2, so the open covering $\{U_a\}_{a \in A}$ of $A$ has a finite subcover, say $\{U_1, U_2, \ldots, U_n\}$. 
Theorem 32.3. Every compact Hausdorff space is normal.

Proof. Let $X$ be a compact Hausdorff space. Let $A$ and $B$ be disjoint closed sets in $X$. By Lemma 26.4, for each $a \in A$, there are disjoint open $U_a$ and $V_a$ with $a \in U_x$ and $B \subset V_x$. Since $A$ is closed and $X$ is Hausdorff, then $A$ is compact by Theorem 26.2, so the open covering $\{U_a\}_{a \in A}$ of $A$ has a finite subcover, say $\{U_1, U_2, \ldots, U_n\}$. Then $U = U_1 \cap U_2 \cap \cdots \cap U_n$ and $V = V_1 \cap V_2 \cap \cdots \cap V_n$ are disjoint open sets where $A \subset U$ and $B \subset V$. That is, $X$ is regular. \qed
Theorem 32.3

Theorem 32.3. Every compact Hausdorff space is normal.

Proof. Let $X$ be a compact Hausdorff space. Let $A$ and $B$ be disjoint closed sets in $X$. By Lemma 26.4, for each $a \in A$, there are disjoint open $U_a$ and $V_a$ with $x \in U_x$ and $B \subset V_x$. Since $A$ is closed and $X$ is Hausdorff, then $A$ is compact by Theorem 26.2, so the open covering $\{U_a\}_{a \in A}$ of $A$ has a finite subcover, say $\{U_1, U_2, \ldots, U_n\}$. Then $U = U_1 \cap U_2 \cap \cdots \cap U_n$ and $V = V_1 \cap V_2 \cap \cdots \cap V_n$ are disjoint open sets where $A \subset U$ and $B \subset V$. That is, $X$ is regular.
Theorem 32.4

Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof. Let $X$ be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in $X$. If $X$ has a largest element and $y$ is that element, then $(x, y]$ is a basis element of $y$ (see the definition of "order topology" in Section 14). If $y$ is not the largest element of $X$, then $(x, y]$ equals the open set $(x, y')$ where $y'$ is the immediate successor of $y$ (since $X$ is well-ordered, every nonempty subset of $X$ has a smallest element and so every element $x \in X$ other than the largest element of $X$ has an immediate successor; namely the smallest element of $\{y \in X | v > x\}$). In either case, $(x, y]$ is open in $X$. Now let $A$ and $B$ be disjoint closed sets in $X$. First, suppose that neither $A$ nor $B$ contains the smallest element $a_0$ of $X$. For each $a \in A$, there is a basis element containing $a$ disjoint from $B$ (since $B$ is closed it contains its limit points by Corollary 17.7, so $a$ is not a limit point of $B$).
Theorem 32.4. Every well-ordered set \( X \) is normal in the order topology.

Proof. Let \( X \) be a well-ordered set. We claim that every interval of the form \( (x, y] \) is open in \( X \). If \( X \) has a largest element and \( y \) is that element, then \( (x, y] \) is a basis element of \( y \) (see the definition of “order topology” in Section 14).
Theorem 32.4

Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof. Let $X$ be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in $X$. If $X$ has a largest element and $y$ is that element, then $(x, y]$ is a basis element of $y$ (see the definition of “order topology” in Section 14). If $y$ is not the largest element of $X$, then $(x, y]$ equals the open set $(x, y')$ where $y'$ is the immediate successor of $y$ (since $X$ is well-ordered, every nonempty subset of $X$ has a smallest element and so every element $x \in X$ other than the largest element of $X$ has an immediate successor; namely the smallest element of $\{y \in X \mid v > x\}$). In either case, $(x, y]$ is open in $X$. 
Theorem 32.4

**Theorem 32.4.** Every well-ordered set $X$ is normal in the order topology.

**Proof.** Let $X$ be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in $X$. If $X$ has a largest element and $y$ is that element, then $(x, y]$ is a basis element of $y$ (see the definition of “order topology” in Section 14). If $y$ is not the largest element of $X$, then $(x, y]$ equals the open set $(x, y')$ where $y'$ is the immediate successor of $y$ (since $X$ is well-ordered, every nonempty subset of $X$ has a smallest element and so every element $x \in X$ other than the largest element of $X$ has an immediate successor; namely the smallest element of $\{y \in X \mid v > x\}$). In either case, $(x, y]$ is open in $X$.

Now let $A$ and $B$ be disjoint closed sets in $X$. First, suppose that neither $A$ nor $B$ contains the smallest element $a_0$ of $X$. 
Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof. Let $X$ be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in $X$. If $X$ has a largest element and $y$ is that element, then $(x, y]$ is a basis element of $y$ (see the definition of “order topology” in Section 14). If $y$ is not the largest element of $X$, then $(x, y]$ equals the open set $(x, y')$ where $y'$ is the immediate successor of $y$ (since $X$ is well-ordered, every nonempty subset of $X$ has a smallest element and so every element $x \in X$ other than the largest element of $X$ has an immediate successor; namely the smallest element of $\{y \in X \mid v > x\}$). In either case, $(x, y]$ is open in $X$.

Now let $A$ and $B$ be disjoint closed sets in $X$. First, suppose that neither $A$ nor $B$ contains the smallest element $a_0$ of $X$. For each $a \in A$, there is a basis element containing $a$ disjoint from $B$ (since $B$ is closed it contains its limit points by Corollary 17.7, so $a$ is not a limit point of $B$).
Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof. Let $X$ be a well-ordered set. We claim that every interval of the form $(x, y]$ is open in $X$. If $X$ has a largest element and $y$ is that element, then $(x, y]$ is a basis element of $y$ (see the definition of “order topology” in Section 14). If $y$ is not the largest element of $X$, then $(x, y]$ equals the open set $(x, y')$ where $y'$ is the immediate successor of $y$ (since $X$ is well-ordered, every nonempty subset of $X$ has a smallest element and so every element $x \in X$ other than the largest element of $X$ has an immediate successor; namely the smallest element of $\{y \in X \mid y > x\}$). In either case, $(x, y]$ is open in $X$.

Now let $A$ and $B$ be disjoint closed sets in $X$. First, suppose that neither $A$ nor $B$ contains the smallest element $a_0$ of $X$. For each $a \in A$, there is a basis element containing $a$ disjoint from $B$ (since $B$ is closed it contains its limit points by Corollary 17.7, so $a$ is not a limit point of $B$).
Theorem 32.4 (continued 1)

Proof (continued). Since $a$ is not the smallest element of $X$, the basis element containing $a$ contains some interval of the form $(x, a]$. For each $a \in A$, choose such an interval $(x_a, a]$ disjoint from set $B$. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from set $A$. Notice that each $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a + 1)$ and $(y_b, b + 1)$ where “+1” represents the immediate successor.

The sets $U = \bigcup_{a \in A} (x_a, a]$ and $V = \bigcup_{b \in B} (y_b, b]$ are open sets where $A \subset U$ and $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. WLOG, $a < b$. If $a \leq y_b$ then the two intervals are disjoint CONTRADICTING the assumption that $z \in (x_a, a] \cap (y_b, b]$. If $a > y_b$ then $y_b < a < b$ and $a \in (y_b, b]$, CONTRADICTING the fact that $(y_b, b]$ is disjoint from $A$. So the assumption that there is $z \in U \cap V$ is false and so $U$ and $V$ are in fact disjoint.
Proof (continued). Since $a$ is not the smallest element of $X$, the basis element containing $a$ contains some interval of the form $(x, a]$. For each $a \in A$, choose such an interval $(x_a, a]$ disjoint from set $B$. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from set $A$. Notice that each $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a + 1)$ and $(y_b, b + 1)$ where “$+1$” represents the immediate successor. The sets

$$U = \bigcup_{a \in A} (x_a, a] \text{ and } V = \bigcup_{b \in B} (y_b, b]$$

are open sets where $A \subset U$ and $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. WLOG, $a < b$. If $a \leq y_b$ then the two intervals are disjoint CONTRADICTING the assumption that $z \in (x_a, a] \cap (y_b, b]$. If $a > y_b$ then $y_b < a < b$ and $a \in (y_b, b)$, CONTRADICTING the fact that $(y_b, b]$ is disjoint from $A$.

So the assumption that there is $z \in U \cap V$ is false and so $U$ and $V$ are in fact disjoint.
Theorem 32.4 (continued 1)

**Proof (continued).** Since \( a \) is not the smallest element of \( X \), the basis element containing \( a \) contains some interval of the form \((x, a]\). For each \( a \in A \), choose such an interval \((x_a, a]\) disjoint from set \( B \). Similarly, for each \( b \in B \), choose an interval \((y_b, b]\) disjoint from set \( A \). Notice that each \((x_a, a]\) and \((y_b, b]\) is open since each is of the form \((x_a, a + 1]\) and \((y_b, b + 1]\) where “+1” represents the immediate successor. The sets

\[
U = \bigcup_{a \in A} (x_a, a]\ and \ V = \bigcup_{b \in B} (y_b, b]\n\]

are open sets where \( A \subset U \) and \( B \subset V \). **ASSUME** \( z \in U \cap V \). Then \( z \in (x_a, a] \cap (y_b, b]\) for some \( a \in A \) and \( b \in B \). WLOG, \( a < b \). If \( a \leq y_b \) then the two intervals are disjoint **CONTRADICTING** the assumption that \( z \in (x_a, a] \cap (y_b, b]\). If \( a > y_b \) then \( y_b < a < b \) and \( a \in (y_b, b]\), **CONTRADICTING** the fact that \((y_b, b]\) is disjoint from \( A \).
Theorem 32.4 (continued 1)

Proof (continued). Since $a$ is not the smallest element of $X$, the basis element containing $a$ contains some interval of the form $(x, a]$. For each $a \in A$, choose such an interval $(x_a, a]$ disjoint from set $B$. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from set $A$. Notice that each $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a]$ and $(y_b, b]$ is open since each is of the form $(x_a, a + 1)$ and $(y_b, b + 1)$ where “+1” represents the immediate successor. The sets

$$U = \bigcup_{a \in A}(x_a, a] \quad \text{and} \quad V = \bigcup_{b \in B}(y_b, b]$$

are open sets where $A \subset U$ and $B \subset V$. ASSUME $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. WLOG, $a < b$. If $a \leq y_b$ then the two intervals are disjoint CONTRADICTING the assumption that $z \in (z_a, a] \cap (y_b, b]$. If $a > y_b$ then $y_b < a < b$ and $a \in (y_b, b]$, CONTRADICTING the fact that $(y_b, b]$ is disjoint from $A$. So the assumption that there is $z \in U \cap V$ is false and so $U$ and $V$ are in fact disjoint.
Theorem 32.4 (continued 1)

Proof (continued). Since \( a \) is not the smallest element of \( X \), the basis element containing \( a \) contains some interval of the form \((x, a]\). For each \( a \in A \), choose such an interval \((x_a, a]\) disjoint from set \( B \). Similarly, for each \( b \in B \), choose an interval \((y_b, b]\) disjoint from set \( A \). Notice that each \((x_a, a]\) and \((y_b, b]\) is open since each is of the form \((x_a, a]\) and \((y_b, b]\) is open since each is of the form \((x_a, a + 1]\) and \((y_b, b + 1]\) where “+1” represents the immediate successor. The sets

\[
U = \bigcup_{a \in A} (x_a, a]\) and \( V = \bigcup_{b \in B} (y_b, b]\)
\]

are open sets where \( A \subset U \) and \( B \subset V \). ASSUME \( z \in U \cap V \). Then \( z \in (x_a, a]\) \( \cap \) \((y_b, b]\) for some \( a \in A \) and \( b \in B \). WLOG, \( a < b \). If \( a \leq y_b \) then the two intervals are disjoint CONTRADICTING the assumption that \( z \in (z_a, a]\) \( \cap \) \((y_b, b]\). If \( a > y_b \) then \( y_b < a < b \) and \( a \in (y_b, b]\), CONTRADICTING the fact that \((y_b, b]\) is disjoint from \( A \). So the assumption that there is \( z \in U \cap V \) is false and so \( U \) and \( V \) are in fact disjoint.
Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof (continued). So the normality condition is satisfied when neither (closed) $A$ nor $B$ contains the smallest element of $X$.

Finally, suppose $A$ and $B$ are disjoint closed sets in $X$ where $A$ contains the smallest element $a_0$ in $X$ where $A$ contains the smallest element $a_0$ of $X$. The set $\{a_0\}$ is both open and closed in $X$, $\{a_0\} = [a_0, a_0 + 1)$ and $X \setminus \{a_0\} = \bigcup_{x \in X} (a_0, x)$. 
Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof (continued). So the normality condition is satisfied when neither (closed) $A$ nor $B$ contains the smallest element of $X$.

Finally, suppose $A$ and $B$ are disjoint closed sets in $X$ where $A$ contains the smallest element $a_0$ in $X$ where $A$ contains the smallest element $a_0$ of $X$. The set $\{a_0\}$ is both open and closed in $X$, $\{a_0\} = [a_0, a_0 + 1)$ and $X \setminus \{a_0\} = \bigcup_{x \in X} (a_0, x)$. By the previous paragraph, there exist disjoint open sets $U$ and $V$, neither containing $a_0$, where $A \setminus \{a_0\} \subset U$ and $B \subset V$ (where $A \setminus \{a_0\}$ and $B$ are closed, disjoint sets). Then $U \cup \{a_0\}$ and $V$ are disjoint open sets containing $A$ and $B$ respectively.
Theorem 32.4. Every well-ordered set $X$ is normal in the order topology.

Proof (continued). So the normality condition is satisfied when neither (closed) $A$ nor $B$ contains the smallest element of $X$.

Finally, suppose $A$ and $B$ are disjoint closed sets in $X$ where $A$ contains the smallest element $a_0$ in $X$, where $A$ contains the smallest element $a_0$ of $X$. The set $\{a_0\}$ is both open and closed in $X$, $\{a_0\} = [a_0, a_0 + 1)$ and $X \setminus \{a_0\} = \bigcup_{x \in X} (a_0, x)$. By the previous paragraph, there exist disjoint open sets $U$ and $V$, neither containing $a_0$, where $A \setminus \{a_0\} \subset U$ and $B \subset V$ (where $A \setminus \{a_0\}$ and $B$ are closed, disjoint sets). Then $U \cup \{a_0\}$ and $V$ are disjoint open sets containing $A$ and $B$ respectively. So the normality condition is satisfied when one of $A$ or $B$ contains the smallest element of $X$. Hence, $X$ is normal.
Theorem 32.4. Every well-ordered set \( X \) is normal in the order topology.

Proof (continued). So the normality condition is satisfied when neither (closed) \( A \) nor \( B \) contains the smallest element of \( X \).

Finally, suppose \( A \) and \( B \) are disjoint closed sets in \( X \) where \( A \) contains the smallest element \( a_0 \) in \( X \) where \( A \) contains the smallest element \( a_0 \) of \( X \). The set \( \{a_0\} \) is both open and closed in \( X \), \( \{a_0\} = [a_0, a_0 + 1) \) and \( X \setminus \{a_0\} = \bigcup_{x \in X} (a_0, x) \). By the previous paragraph, there exist disjoint open sets \( U \) and \( V \), neither containing \( a_0 \), where \( A \setminus \{a_0\} \subset U \) and \( B \subset V \) (where \( A \setminus \{a_0\} \) and \( B \) are closed, disjoint sets). Then \( U \cup \{a_0\} \) and \( V \) are disjoint open sets containing \( A \) and \( B \) respectively. So the normality condition is satisfied when one of \( A \) or \( B \) contains the smallest element of \( X \). Hence, \( X \) is normal.