Lemma 37.1 (continued 1)

\[ \forall A \subseteq \mathcal{P}(X) \; \exists \bigcap_{A \subseteq X} B \subseteq \bigcup_{A \subseteq X} B \]

\[ \forall A \subseteq \mathcal{P}(X) \; \exists \bigcup_{A \subseteq X} B \subseteq \bigcap_{A \subseteq X} B \]

\[ \forall A \subseteq \mathcal{P}(X) \; \exists \bigcap_{A \subseteq X} B \subseteq \bigcup_{A \subseteq X} B \]

\[ \forall A \subseteq \mathcal{P}(X) \; \exists \bigcup_{A \subseteq X} B \subseteq \bigcap_{A \subseteq X} B \]

(continued)
The finite intersection property (FIP) states that if a collection of sets has the finite intersection property, then any finite subcollection of these sets also has the finite intersection property.

**Theorem 3.2**: The FIP is a topological property.

**Proof**: Let \( X = \prod_{i \in I} X_i \) where each \( X_i \) is compact. Let \( A \) be a collection of compact subsets of \( X \) which is maximal with respect to the finite intersection property.

By Lemma 3.2, \( A \) is an element of \( D \) and so \( a \subseteq D \) and so \( A = A \). Let \( \{ B \} \) be a set. Let \( a \) be a set. Let \( a \) be a collection of subsets of \( X \) that is maximal with respect to the finite intersection property.

**Lemma 3.2**: Let \( X \) be a set. Let \( A \) be a set. Let \( a \) be a collection of subsets of \( X \) that is maximal with respect to the finite intersection property.

The finite intersection property of \( A \) is maximal with respect to the finite intersection property of \( A \). Therefore, \( A = A \). Let \( \{ B \} \) be a set. Let \( a \) be a collection of subsets of \( X \) that is maximal with respect to the finite intersection property.

**Proof**: Let \( \{ B \} \) be a set. Let \( a \) be a collection of subsets of \( X \) that is maximal with respect to the finite intersection property.

By definition of \( \{ B \} \), \( a \subseteq D \) for all \( a \in A \). So \( Z \) is a set. Lemma 3.2. There is \( D \subseteq \{ B \} \) that has the finite intersection property.

The finite intersection property of \( D \) is maximal with respect to the finite intersection property of \( D \). Therefore, \( D = D \). Let \( \{ B \} \) be a set. Let \( a \) be a collection of subsets of \( X \) that is maximal with respect to the finite intersection property.

**Lemma 3.2 (continued)**

The finite intersection property of \( D \) is maximal with respect to the finite intersection property of \( D \). Therefore, \( D = D \). Let \( \{ B \} \) be a set. Let \( a \) be a collection of subsets of \( X \) that is maximal with respect to the finite intersection property.
element containing $x$ belongs to $D$.

$\square$

By Theorem 2.9, $C = \prod_{a \in \mathcal{A}} X_a$ is compact. For every $a \in \mathcal{A}$, $D$ is compact. By Lemma 37.2(a), every subspace intersects every $x \in X_a$ of the previous paragraph. Since $x \in X_a$ is a neighborhood of $x_0$ in $X_a$, there is a neighborhood of $x_0$ in $X_a$ compact. Therefore, by Lemma 37.2(a), every basis element containing $x$ belongs to $D$. Since $D$ is the finite intersection property, every basis element containing $x$ belongs to $D$. By Theorem 37.2, $\prod_{a \in \mathcal{A}} X_a$ of Theorem 2.9 (see page 114). Let $(\beta_n)_{n=1}^{\infty}$ be a sequence of elements of the form $(\gamma_n, \ldots, \gamma_2, \gamma_1)$ where $\gamma_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ (see page 114). For some $\gamma_0 \in \mathcal{A}$, for all $n \in \mathbb{N}$.

# Proof (continued)

Now every basis element of the product topology is of the form $\prod_{a \in \mathcal{A}} U_a$ where $U_a \subseteq X_a$ is open in $X_a$. We will show that $x \in X_a$ is compact if and only if $x \in X_a$ is compact. Since $x \in X_a$ is compact, by Theorem 37.2, $\prod_{a \in \mathcal{A}} X_a$ of Theorem 2.9 (see page 114). Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of elements of the form $(\delta_n, \ldots, \delta_2, \delta_1)$ where $\delta_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ (see page 114). For some $\delta_0 \in \mathcal{A}$, for all $n \in \mathbb{N}$.

# Proof (continued)

Consider $\prod_{a \in \mathcal{A}} U_a$, where $U_a \subseteq X_a$ is open in $X_a$. Since $X_a$ is the finite intersection property, the collection $D$ does not contain this collection. Hence, $D$ is the finite intersection property. Notice that this collection has the finite intersection property because $D$ does.