Chapter 6. Metrization Theorems and Paracompactness
Section 39. Local Finiteness—Proofs of Theorems
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Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of $X$. Then:

(a) Any subcollection of $\mathcal{A}$ is locally finite.
(b) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.
(c) $\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} \overline{A}$.

Proof. (a) This follows trivially from the definition.
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(b) The collection \( \mathcal{B} = \{ \overline{A} \}_{A \in \mathcal{A}} \) of the closures of the elements of \( \mathcal{A} \) is locally finite.

(c) \( \bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} \overline{A} \).

Proof. (a) This follows trivially from the definition.

(b) First, note that any open set \( U \) that intersects set \( \overline{A} \) must also intersect \( A \) (since \( \overline{A} = A \cup A' \) where \( A' \) is the set of limit points of \( A \), by Theorem 17.6).
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(b) First, note that any open set $U$ that intersects set $\overline{A}$ must also intersect $A$ (since $\overline{A} = A \cup A'$ where $A'$ is the set of limit points of $A$, by Theorem 17.6). So if $U$ is a neighborhood of $x \in X$ that only intersects finitely many $A \in \mathcal{A}$, say $A_1, A_2, \ldots, A_n$, then $U$ also only intersects $\overline{A_1}, \overline{A_2}, \ldots, \overline{A_n} \in \mathcal{B}$ (see Theorem 17.5(a); it could be that $\overline{A_i} = \overline{A_j}$ and $U$ could actually intersect fewer elements of $\mathcal{B}$ than of $\mathcal{A}$).
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(c) $\bigcup_{A \in \mathcal{A}} \overline{A} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

Proof (continued). (c) Denote $Y = \bigcup_{A \in \mathcal{A}} A$. Now each $A \in \mathcal{A}$ is a subset of $Y$ so $\overline{A} \subset \overline{Y}$ (apply Theorem 17.5(a), say). Now let $x \in \overline{Y}$ and let $U$ be a neighborhood of $x$. Then, since $\mathcal{A}$ is locally finite in $X$, $U$ intersects only finitely many elements of $\mathcal{A}$, say $A_1, A_2, \ldots, A_k$. Assume $x \notin \overline{A}_1, x \notin \overline{A}_2, \ldots, x \notin \overline{A}_k$. Then set $\left(\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_k\right)$ is a neighborhood of $x$ that intersects no element of $\mathcal{A}$. But then $U$ is a neighborhood of $x$ that does not intersect $Y = \bigcup_{A \in \mathcal{A}} A$, a CONTRADICTION to the fact that $x \in \overline{Y}$ (see Theorem 17.5(a)).
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Lemma 39.2

**Lemma 39.2.** Let $X$ be a metrizable space. If $\mathcal{A}$ is an open covering of $X$, then there is an open covering $\mathcal{E}$ of $X$ refining $\mathcal{A}$ that is countable locally finite.

**Proof.** We will use the Well-Ordering Theorem: “If $A$ is a set, there exists an order relation on $A$ that is a well-ordering.” Recall that this is equivalent to the Axiom of Choice. Let $<$ be a well-ordering for set $\mathcal{A}$. 
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Since $X$ is metrizable, there is a metric $d$ on $X$. Let $n \in \mathbb{N}$. Given $U \in \mathcal{A}$, define $S_n(U)$ as the subset of $U$ obtained by “shrinking” $U$ a distance of $1/n$: $S_n(U) = \{x \mid B(x,1/n) \subset U\}$. 
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Lemma 39.2 (continued 1)

Proof (continued). Let $V, W \in \mathcal{A}$ with $V \neq W$. If $x \in T_n(V)$ and $y \in T_n(W)$ then we claim $d(x, y) \geq 1/n$ (see Figure 39.1 in which $U < V < W$).

To justify this, say $V < W$. Since $x \in T_n(V) \subset S_n(V)$, then the $1/n$-neighborhood of $x$ lies in $V$ (by the definition of $S_n(V)$). Since $V < W$ and $y \in T_n(V)$ then $y \notin V$ (by the definition of $T_n(W)$), and so $y$ is not in the $1/n$-neighborhood of $x$. 
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Lemma 39.2 (continued 2)

Proof (continued). Now for each \( U \in \mathcal{A} \), define
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E_n(U) = \{ B(x, 1/(3n)) \mid x \in T_n(U) \}
\]
where \( B(x, 1/(3n)) = \{ y \in X \mid d(x, y) < 1/(3n) \} \). That is, \( E_n(U) \) is an “expansion” of \( T_n(U) \) by an amount of \( 1/(3n) \). Notice that \( E_n(U) \subset U \) and since \( E_n(U) \) is a union of “open balls” then \( E_n(U) \) itself is open.
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![Figure 39.2](image-url)
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Lemma 39.2 (continued 3)

**Proof (continued).** By the construction of $E_n(V)$ and $E_n(W)$, there are $x' \in T_n(V)$ and $y' \in T_n(W)$ such that $d(x, x') \leq 1/(3n)$ and $d(y, y') \leq 1/(3n)$. As observed above, $d(x', y') \geq 1/n$ for such $x$ and $y$. So

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\frac{1}{n} \leq d(x', y') \leq d(x', x) + d(x, y) + d(y, y') \text{ by the Triangle Inequality}
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or $d(x, y) \geq 1/(3n)$. 

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or $d(x, y) \geq 1/(3n)$.

Now define $\mathcal{E}_n = \{E_n(U) \mid U \in \mathcal{A}\}$. We claim that $\mathcal{E}_n$ is a locally finite collection of open sets that refines $\mathcal{S}$. First, by construction, each $E_n(U)$ is open and refines $\mathcal{A}$ since $E_n(U) \subset U$ for all $U \in \mathcal{A}$. 
Lemma 39.2 (continued 3)

**Proof (continued).** By the construction of \( E_n(V) \) and \( E_n(W) \), there are \( x' \in T_n(V) \) and \( y' \in T_n(W) \) such that \( d(x, x') \leq 1/(3n) \) and \( d(y, y') \leq 1/(3n) \). As observed above, \( d(x', y') \geq 1/n \) for such \( x \) and \( y \). So

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or \( d(x, y) \geq 1/(3n) \).

Now define \( \mathcal{E}_n = \{ E_n(U) \mid U \in \mathcal{A} \} \). We claim that \( \mathcal{E}_n \) is a locally finite collection of open sets that refines \( \mathcal{S} \). First, by construction, each \( E_n(U) \) is open and refines \( \mathcal{A} \) since \( E_n(U) \subset U \) for all \( U \in \mathcal{A} \). For any \( x \in X \), the \( 1/(6n) \)-neighborhood of \( x \) intersects at most one element of \( \mathcal{E}_n \) (since the elements of \( \mathcal{E}_n \) are a distance of at least \( 1/(3n) \) apart). So \( \mathcal{E}_n \) is locally finite.
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Proof (continued). By the construction of $E_n(V)$ and $E_n(W)$, there are $x' \in T_n(V)$ and $y' \in T_n(W)$ such that $d(x, x') \leq 1/(3n)$ and $d(y, y') \leq 1/(3n)$. As observed above, $d(x', y') \geq 1/n$ for such $x$ and $y$. So

$$\frac{1}{n} \leq d(x', y') \leq d(x', x) + d(x, y) + d(y, y')$$

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Lemma 39.2. Let $X$ be a metrizable space. If $\mathcal{A}$ is an open covering of $X$, then there is an open covering $\mathcal{E}$ of $X$ refining $\mathcal{A}$ that is countable locally finite.

Proof (continued). Now $\mathcal{E}_n$ may not cover $X$ for any given $n \in \mathbb{N}$ (see Figure 39.2), so consider $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$. Let $x \in X$. We hypothesized that $\mathcal{A}$ was a covering of $X$, so use the well-ordering on $\mathcal{A}$ to choose $U$ as the “first” (that is, $<$-least) element of $\mathcal{A}$ that contains $x$. Since $U$ is open (by hypothesis), there is some $n \in \mathbb{N}$ such that $B(x, 1/n) \subset U$ (since the topology on $X$ is hypothesized to be the metric topology under metric $d$). Then by the definition of $S_n(U)$, $x \in S_n(U)$. 
Lemma 39.2. Let $X$ be a metrizable space. If $A$ is an open covering of $X$, then there is an open covering $\mathcal{E}$ of $X$ refining $A$ that is countable locally finite.

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Since each $\mathcal{E}_n$ is a refinement of $\mathcal{A}$ then $\mathcal{E}$ is a refinement of $\mathcal{A}$ and since each $\mathcal{E}_n$ is locally finite, then $\mathcal{E}$ is countably locally finite, as claimed. $\square$
Lemma 39.2 (continued 4)

**Lemma 39.2.** Let $X$ be a metrizable space. If $\mathcal{A}$ is an open covering of $X$, then there is an open covering $\mathcal{E}$ of $X$ refining $\mathcal{A}$ that is countable locally finite.

**Proof (continued).** Now $\mathcal{E}_n$ may not cover $X$ for any given $n \in \mathbb{N}$ (see Figure 39.2), so consider $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$. Let $x \in X$. We hypothesized that $\mathcal{A}$ was a covering of $X$, so use the well-ordering on $\mathcal{A}$ to choose $U$ as the “first” (that is, $<\text{-}least$) element of $\mathcal{A}$ that contains $x$. Since $U$ is open (by hypothesis), there is some $n \in \mathbb{N}$ such that $B(x, 1/n) \subset U$ (since the topology on $X$ is hypothesized to be the metric topology under metric $d$). Then by the definition of $S_n(U)$, $x \in S_n(U)$. Since $U$ is the “first” element of $\mathcal{A}$ that contains $x$, then by the definition of $T_n(U)$ we have $x \in T_n(U)$. Since $T_n(U) \subset E_n(U)$, then $x \in E_n(U)$. Therefore, $\mathcal{E}$ is a covering of $X$. Since each $\mathcal{E}_n$ is a refinement of $\mathcal{A}$ then $\mathcal{E}$ is a refinement of $\mathcal{A}$ and since each $\mathcal{E}_n$ is locally finite, then $\mathcal{E}$ is countably locally finite, as claimed. \qed