Lemma 40.1 (continued 1)

**Proof.** We follow Munkres' three-step proof.

1. **Step 1.** Let \( W \) be an open set in \( X \). We claim there is a countable family of \( \mathcal{G} \)-sets that are disjoint open sets with \( C \cap \bigcup(W \mathcal{G}) = \emptyset \).

2. **Step 2.** Let \( \mathcal{G} \) be a \( \mathcal{G} \)-family of disjoint open sets in \( X \).

3. **Step 3.** We now show that \( \mathcal{G} \) is a \( \mathcal{G} \)-family.

Lemma 40.1 (continued 2)

**Proof.** Applying the regularity to point \( x \) and closed set \( \overline{C} \), the set \( X \setminus \overline{C} \) is regular. Since \( \overline{C} \) is closed, we can find \( \mathcal{G} \)-sets \( G_n \), one for each \( x \in \overline{C} \), such that \( x \in G_n \) and \( G_n \cap \overline{C} = \emptyset \). Since \( \bigcup G_n = \overline{C} \), we can choose \( \mathcal{G} \)-sets \( G_n \) such that \( \bigcup G_n = \overline{C} \). Since each \( G_n \) is locally finite, the union of countably many \( \mathcal{G} \)-sets is locally finite.

**Theorem.** The Nagata-Smitsuji Theorem. Let \( X \) be a regular space with a basis \( \mathcal{B} \) that is countably locally finite. Then \( X \) is normal.

**Proof.** Since \( X \) is normal, every closed set in \( X \) is a \( G_\delta \)-set in \( X \).
The intersection of these finite number of neighborhoods of $x_0$. Let $V$ be the intersection of all these neighborhoods. Since $V$ is an open set in $\mathbb{R}$ and $x_0 \notin V$, there is an $\varepsilon > 0$ such that $B(x_0, \varepsilon) \not\subseteq V$. Let $V' = B(x_0, \varepsilon) \setminus V$. Then $V'$ is an open set in $\mathbb{R}$ and $x_0 \not\in V'$. We have shown that for each $x \in X$ there is a basis element $B(x, \varepsilon)$ such that $B(x, \varepsilon) \subseteq V$ or $B(x, \varepsilon) \subseteq V'$.

Notice that the union of all $B(x, \varepsilon)$ for $x \in X$ is $X$. Therefore, $\mathcal{B}$ is a basis for $X$.

Proof (continued): To show that $\mathcal{B}$ is a basis for $X$, we need to prove that if $U$ is any open set in $X$, then $U = \bigcup_{B \in \mathcal{B}} B$. Let $U$ be any open set in $X$. Then $U$ is the union of all basis elements $B \subseteq U$. Thus $U = \bigcup_{B \in \mathcal{B}} B$.

Theorem 4.3 (continued): The Nagerman-Minh-Dong Metrization Theorem

Lemma 4.2. Let $x_0 \in X$. Then $x_0$ is a limit point of $X$ if and only if $x_0$ is not in a closed set.

Theorem 4.3. The Nagerman-Minh-Dong Metrization Theorem

Lemma 4.3. The Nagerman-Minh-Dong Metrization Theorem

Theorem 4.4. The Nagerman-Minh-Dong Metrization Theorem

Proof. First, assume $X$ is regular with a countably locally finite basis. Then each compact subset of $X$ can be covered by a countable family of basis elements.

Theorem 4.5. The Nagerman-Minh-Dong Metrization Theorem

Lemma 4.4. The Nagerman-Minh-Dong Metrization Theorem

Theorem 4.6. The Nagerman-Minh-Dong Metrization Theorem

Proof. This was shown in Section 33. Exercise 33.4. We prove it now.

Theorem 4.7. The Nagerman-Minh-Dong Metrization Theorem

Lemma 4.5. The Nagerman-Minh-Dong Metrization Theorem

Theorem 4.8. The Nagerman-Minh-Dong Metrization Theorem

Proof. This was shown in Section 34. Exercise 34.4. We prove it now.

Theorem 4.9. The Nagerman-Minh-Dong Metrization Theorem

Lemma 4.6. The Nagerman-Minh-Dong Metrization Theorem

Theorem 4.10. The Nagerman-Minh-Dong Metrization Theorem

Proof. This was shown in Section 35. Exercise 35.4. We prove it now.
Theorem 4.3 (continued)

Proof (continued). Given $x \in X$ and $\epsilon > 0$, there is some $m \in \mathbb{N}$ with $1/m < \epsilon/2$. There is some open covering of $X$ by definition of $B^m$, there is some open covering of $X$ by definition of $B^m$, there is some open covering of $X$ by definition of $B^m$.

A topological space $X$ is metrizable if and only if $X$ is regular and has a basis that is countably locally finite. A topological space $X$ is metrizable if and only if $X$ is regular and has a basis that is countably locally finite.

Theorem 4.3 (The Nagata-Smirnov Metrization Theorem)

Proof (continued). Now suppose $X$ is metrizable. Then $X$ is normal by Theorem 3.2 and therefore is regular (since every normal space is regular).

Therefore $\exists \epsilon > 0$ such that $B^m \cap \{x \in X^m \mid \langle x \rangle < \epsilon \}$ is a neighborhood of $x$. By Lemma 3.9, 2.3 open balls of radius $1/m$ of any $x \in X^m$ is a neighborhood of $x$.

Now to show $X$ has a basis that is countably locally finite. A basis for $X$ is regular and so $X$ is metrizable. Then $\exists \epsilon > 0$ such that $B^m \cap \{x \in X^m \mid \langle x \rangle < \epsilon \}$ is a neighborhood of $x$. By Lemma 3.9, 2.3 open balls of radius $1/m$ of any $x \in X^m$ is a neighborhood of $x$.

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