Lemma 45.2. A metric space $(X, d)$ is compact if and only if it is sequentially compact.

Proof. Suppose $X$ is sequentially compact. Then every sequence in $X$ has a convergent subsequence, since $X$ is a Cauchy-complete metric space. Therefore, for each $n$, there exists a convergent subsequence $(a_{n+1})$ of $(a_n)$, which converges to a point $x_n$. Now, choose $n < k$ such that $k - n > 2$. Now, for $j > k$, the points $a_j$ and $a_k$ are contained in ball $B_r(1/n, 1/n)$ and lie in $X_n$, where $X_n$ is the finite subset of $X_n$.

Section 45. Compactness in Metric Spaces—Proofs of Theorems
Lemma 4.3 (continued)

Let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be the collection of all functions $f: \mathcal{X} \rightarrow \mathcal{Y}$. Let $\mathcal{X}$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.

Lemma 4.3 (continued)

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Lemma 4.3 (continued)

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Lemma 4.3 (continued)

Let $\mathcal{X}$ be a topological space and let $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ be an arbitrary point of $\mathcal{X}$. Since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$, there is a corresponding neighborhood of $\mathcal{X}$, for any $\alpha$, there is a corresponding neighborhood of $\mathcal{X}$.

Therefore, $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$ since $\mathcal{X}$ is an arbitrary point of $\mathcal{X}$.
Theorem 4.1.4 (continued 3)

Theorem 4.1.4 (continued 2)

Theorem 4.4.2

Theorem 4.4.3

Theorem 4.4.4 (continued 1)

Theorem 4.4.4 (continued 2)

Theorem 4.5.4. The Classical Version of Ascoli's Theorem.
Theorem 4.2: If $\mathcal{F}$ is a compact, complete, and totally bounded subset of $X$, then $\mathcal{F}$ is compact.

Proof (continued). By Step 2, $\mathcal{F}$ is equicontinuous and pointwise bounded under $d$. Since $\mathcal{F}$ is compact, $\mathcal{F}$ is bounded. Therefore, $\mathcal{F}$ is also equicontinuous.

Thus, by Theorem 2.1, $\mathcal{F}$ is compact.

Corollary 4.5: Let $X$ be a compact space. Then $\mathcal{F}$ is a closed subset of $X$.

Proof. By the Heine-Borel Theorem, $\mathcal{F}$ is the desired compact subset of $X$.

For all $x \in X$, let $\mathcal{F} = (I + N)0 \mathcal{F}$. Then $\mathcal{F}$ is closed, since it is the uniform closure of the set $\{x \in X : \langle x, y \rangle \leq 0\}$ for all $y \in \mathcal{F}$.

Since $\mathcal{F}$ is bounded, there exists a covering of $\mathcal{F}$ by open sets $U_i$ such that $\mathcal{F}$ is covered by finitely many $U_i$.

Therefore, $\mathcal{F}$ is compact.

Conversely, if $\mathcal{F}$ is closed, then $\mathcal{F} = \mathcal{F}$, and $\mathcal{F}$ is bounded under $d$. Thus, $\mathcal{F}$ is compact, and $\mathcal{F}$ is equicontinuous. 

Theorem 4.4: If $\mathcal{F}$ is a compact, complete, and totally bounded subset of $X$, then $\mathcal{F}$ is equicontinuous.

Proof. By Theorem 2.1, $\mathcal{F}$ is compact. Therefore, $\mathcal{F}$ is bounded. Since $\mathcal{F}$ is compact, $\mathcal{F}$ is equicontinuous.

Corollary 4.5: Let $X$ be a compact space. Then $\mathcal{F}$ is closed.

Proof. By the Heine-Borel Theorem, $\mathcal{F}$ is the desired compact subset of $X$.

For all $x \in X$, let $\mathcal{F} = (I + N)0 \mathcal{F}$. Then $\mathcal{F}$ is closed, since it is the uniform closure of the set $\{x \in X : \langle x, y \rangle \leq 0\}$ for all $y \in \mathcal{F}$.

Since $\mathcal{F}$ is bounded, there exists a covering of $\mathcal{F}$ by open sets $U_i$ such that $\mathcal{F}$ is covered by finitely many $U_i$.

Therefore, $\mathcal{F}$ is compact.

Conversely, if $\mathcal{F}$ is closed, then $\mathcal{F} = \mathcal{F}$, and $\mathcal{F}$ is bounded under $d$. Thus, $\mathcal{F}$ is compact, and $\mathcal{F}$ is equicontinuous.