Chapter 9. The Fundamental Group
Section 58. Deformation Retracts and Homotopy Type—Proofs of Theorems
Lemma 58.1. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If $h$ and $k$ are homotopic and if the image of the base point $x_0 \in X$ remains fixed at $y_0$ during the homotopy then the induced homomorphisms $h_*$ and $k_*$ are equal.

**Proof.** Let $H : X \times I \to Y$ be a homotopy between $h$ and $k$ such that $H(x_0, t) = y_0$ for all $t \in I$. Let $f$ be a loop in $X$ based at $x_0$. Consider the composition

$$I \times I \xrightarrow{f \times \text{ID}} X \times I \xrightarrow{H} Y$$

(1)

Now $H(x, 0) = h(x)$ and $H(x, 1) = k(x)$ by the definition of homotopy. Also, $(f \times \text{ID})(x, 0) = (f(x), 0)$ and $(f \times \text{ID})(x, 1) = (f(x), 1)$ so $H \circ (f \times \text{ID})(x, 0) = h(f(x))$ and $H \circ (f \times \text{ID})(x, 1) = k(f(x))$. 
Lemma 58.1. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If $h$ and $k$ are homotopic and if the image of the base point $x_0 \in X$ remains fixed at $y_0$ during the homotopy then the induced homomorphisms $h_\ast$ and $k_\ast$ are equal.

Proof. Let $H : X \times I \to Y$ be a homotopy between $h$ and $k$ such that $H(x_0, t) = y_0$ for all $t \in I$. Let $f$ be a loop in $X$ based at $x_0$. Consider the composition

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Lemma 58.1. Let \( h, k : (X, x_0) \to (Y, y_0) \) be continuous maps. If \( h \) and \( k \) are homotopic and if the image of the base point \( x_0 \in X \) remains fixed at \( y_0 \) during the homotopy then the induced homomorphisms \( h_* \) and \( k_* \) are equal.

**Proof.** Let \( H : X \times I \to Y \) be a homotopy between \( h \) and \( k \) such that \( H(x_0, t) = y_0 \) for all \( t \in I \). Let \( f \) be a loop in \( X \) based at \( x_0 \). Consider the composition

\[
I \times I \xrightarrow{f \times \text{ID}} X \times I \xrightarrow{H} Y \quad (1)
\]

Now \( H(x, 0) = h(x) \) and \( H(x, 1) = k(x) \) by the definition of homotopy. Also, \( (f \times \text{ID})(x, 0) = (f(x), 0) \) and \( (f \times \text{ID})(x, 1) = (f(x), 1) \) so

\[
H \circ (f \times \text{ID})(x, 0) = h(f(x)) \quad \text{and} \quad H \circ (f \times \text{ID})(x, 1) = k(f(x)).
\]

So \( H \circ (f \times \text{ID}) \) is a homotopy from \( h(f(x)) = h \circ f \) and \( k(f(x)) = k \circ f \).
Lemma 58.1 Continued

Since $f$ is a loop based at $x_0$, then $h \circ f$ and $k \circ f$ are loops based at $y_0$ and since $H$ maps $\{x_0\} \times I$ to $y_0$, the homotopy is a path homotopy. So,

\[ h_\ast[f] = [h \circ f] = [k \circ f] \text{ because of the path homotopy} \]

for all loops $f$ based at $x_0$. So $h_\ast = k_\ast$ on $\pi_1(X, x_0)$.
Since $f$ is a loop based at $x_0$, then $h \circ f$ and $k \circ f$ are loops based at $y_0$ and since $H$ maps $\{x_0\} \times I$ to $y_0$, the homotopy is a path homotopy. So,

$$h_*[f] = [h \circ f]$$
$$= [k \circ f] \text{ because of the path homotopy}$$
$$= k_*[f]$$

for all loops $f$ based at $x_0$. So $h_* = k_*$ on $\pi_1(X, x_0)$. \qed
Theorem 58.2. The inclusion map $j : S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ induces an isomorphism of fundamental groups $\pi_1(S^n, x_0)$ and $\pi_1(\mathbb{R}^{n+1} \setminus \{\vec{0}\}, x_0)$ for $n \geq 1$.

Proof. Let $X = \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ and let $b_0 = (1, 0, \ldots, 0)$. Define $r : X \to S^n$ as $r(\vec{x}) = \vec{x}/\|\vec{x}\|$. Then $r \circ j$ is the identity map on $S^n$ and so $(r \circ j)_* = r_* \circ j_*$ is the identity homomorphism on $\pi_1(S^n, b_0)$. 
Theorem 58.2. The inclusion map \( j : S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\} \) induces an isomorphism of fundamental groups \( \pi_1(S^n, x_0) \) and \( \pi_1(\mathbb{R}^{n+1} \setminus \{\vec{0}\}, x_0) \) for \( n \geq 1 \).

**Proof.** Let \( X = \mathbb{R}^{n+1} \setminus \{\vec{0}\} \) and let \( b_0 = (1, 0, \ldots, 0) \). Define \( r : X \to S^n \) as \( r(\vec{x}) = \vec{x}/\|\vec{x}\| \). Then \( r \circ j \) is the identity map on \( S^n \) and so \( (r \circ j)_* = r_* \circ j_* \) is the identity homomorphism on \( \pi_1(S^n, b_0) \).

Now consider \( j \circ r : x \to x \):
Theorem 58.2. The inclusion map \( j : S^n \to \mathbb{R}^{n+1}\setminus\{\vec{0}\} \) induces an isomorphism of fundamental groups \( \pi_1(S^n, x_0) \) and \( \pi_1(\mathbb{R}^{n+1}\setminus\{\vec{0}\}, x_0) \) for \( n \geq 1 \).

Proof. Let \( X = \mathbb{R}^{n+1}\setminus\{\vec{0}\} \) and let \( b_0 = (1, 0, \ldots, 0) \). Define \( r : X \to S^n \) as \( r(\vec{x}) = \vec{x}/||\vec{x}|| \). Then \( r \circ j \) is the identity map on \( S^n \) and so \( (r \circ j)_* = r_* \circ j_* \) is the identity homomorphism on \( \pi_1(S^n, b_0) \).

Now consider \( j \circ r : X \to X \):

\[
X \xrightarrow{r} S^n \xrightarrow{j} X
\] (3)
Now $j \circ r$ is obviously not the identity map on $X$. But $j \circ r$ is straight line homotopic to the identity on $X$ since

$$H(\vec{x}, t) = (1 - t)\vec{x} + t\vec{x}/\|\vec{x}\|$$

(4)

is a homotopy between the identity map on $X$ (When $t = 0$) and $j \circ r$ (when $t = 1$).

Notice $H(x, t)$ is never $\vec{0}$ since $(1 - t) + t\|\vec{x}\|$ is a number between 1 and $1/\|\vec{x}\|$. Also, $b_0 = (1, 0, ..., 0)$ remains fixed during the homotopy since $\|b_0\| = 1$. 
Theorem 58.2 Continued

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Theorem 58.2 Continued

Now $j \circ r$ is obviously not the identity map on $X$. But $j \circ r$ is straight line homotopic to the identity on $X$ since

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Now $j \circ r$ is obviously not the identity map on $X$. But $j \circ r$ is straight line homotopic to the identity on $X$ since

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Notice $H(x, t)$ is never $\vec{0}$ since $(1 - t) + t\|\vec{x}\|$ is a number between $1$ and $1/\|\vec{x}\|$. Also, $b_0 = (1, 0, ..., 0)$ remains fixed during the homotopy since $\|b_0\| = 1$. By Lemma 58.1, the induced homomorphism $(j \circ r)_* = j_* \circ r_*$ is the identity homomorphism from $\pi_1(X, b_0)$ to itself. So $j_*$ and $r_*$ are invertible and hence one to one and onto (bijective). So $j_*$ induces an isomorphism from $\pi_1(S^n, b_0)$ to $\pi_1(X, b_0)$. \qed
Lemma 58.4. Let $h, k : X \to Y$ be continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If $h$ and $k$ are homotopic, there is a path $\alpha$ in $Y$ from $y_0$ to $y_1$ such that $k_* = \hat{\alpha} \circ h_*$ (where $k_*$ and $h_*$ are induced homomorphisms on the fundamental group and $\hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is defined as $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$—see Figure Below). Indeed, if $H : X \times I \to Y$ is the homotopy between $h$ and $k$, then $\alpha$ is the path $\alpha(t) = H(x_0, t)$. 

![Diagram](image-url)
Lemma 58.4

**Proof.** Let $f$ be an arbitrary loop in $X$ based at $x_0$. We must show that

$$k_*(\lbrack f \rbrack) = \hat{\alpha}(h_*(\lbrack f \rbrack))$$

(5)

By definition of the induced homomorphism, their equation is equivalent to $[k \circ f] = \hat{\alpha}([h \circ f]) = [\bar{\alpha}] \ast [h \circ f] \ast [\alpha]$ by the definition of $\hat{\alpha}$, where $\bar{\alpha}$ is the reverse of $\alpha$. 


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$$[\alpha] \ast [k \circ f] = [h \circ f] \ast [\alpha] \quad (*) \quad (6)$$
Lemma 58.4

**Proof.** Let $f$ be an arbitrary loop in $X$ based at $x_0$. We must show that

$$k_*([f]) = \hat{\alpha}(h_*([f])) \quad (5)$$

By definition of the induced homomorphism, their equation is equivalent to

$$[k \circ f] = \hat{\alpha}([h \circ f]) = [\bar{\alpha}] \ast [h \circ f] \ast [\alpha]$$

by the definition of $\hat{\alpha}$, where $\bar{\alpha}$ is the reverse of $\alpha$. Since $[\alpha] \ast [\bar{\alpha}]$ is the identity, the equation becomes

$$[\alpha] \ast [k \circ f] = [h \circ f] \ast [\alpha] \quad (*) \quad (6)$$

Consider the loops $f_0$ and $f_1$ in the space $X \times I$ given as

$$f_0(s) = (f(s), 0) \text{ and } f_1(s) = (f(s), 1) \quad (7)$$
Lemma 58.4

**Proof.** Let $f$ be an arbitrary loop in $X$ based at $x_0$. We must show that

\[ k_*(\langle f \rangle) = \hat{\alpha}(h_*(\langle f \rangle)) \quad (5) \]

By definition of the induced homomorphism, their equation is equivalent to $[k \circ f] = \hat{\alpha}([h \circ f]) = [\bar{\alpha}] \ast [h \circ f] \ast [\alpha]$ by the definition of $\hat{\alpha}$, where $\bar{\alpha}$ is the reverse of $\alpha$. Since $[\alpha] \ast [\bar{\alpha}]$ is the identity, the equation becomes

\[ [\alpha] \ast [k \circ f] = [h \circ f] \ast [\alpha] \quad (*) \quad (6) \]

Consider the loops $f_0$ and $f_1$ in the space $X \times I$ given as

\[ f_0(s) = (f(s), 0) \text{ and } f_1(s) = (f(s), 1) \quad (7) \]

Consider the path $c$ in $X \times I$ given by

\[ c(t) = (x_0, t) \quad (8) \]
Lemma 58.4

**Proof.** Let $f$ be an arbitrary loop in $X$ based at $x_0$. We must show that

$$k_*(\{f\}) = \hat{\alpha}(h_*(\{f\}))$$  \hspace{1cm} (5)

By definition of the induced homomorphism, their equation is equivalent to $[k \circ f] = \hat{\alpha}([h \circ f]) = [\bar{\alpha}] \ast [h \circ f] \ast [\alpha]$ by the definition of $\hat{\alpha}$, where $\bar{\alpha}$ is the reverse of $\alpha$. Since $[\alpha] \ast [\bar{\alpha}]$ is the identity, the equation becomes

$$[\alpha] \ast [k \circ f] = [h \circ f] \ast [\alpha]$$  \hspace{1cm} (6)

Consider the loops $f_0$ and $f_1$ in the space $X \times I$ given as

$$f_0(s) = (f(s), 0) \text{ and } f_1(s) = (f(s), 1)$$  \hspace{1cm} (7)

Consider the path $c$ in $X \times I$ given by

$$c(t) = (x_0, t)$$  \hspace{1cm} (8)
Lemma 58.4 Continued

Since $H$ is a homotopy between $h$ and $k$, then

$$(†) \quad H \circ f_0 = H(f(s), 0) = h(f(s)) = h \circ f \quad \text{and}$$

$$(††) \quad H \circ f_1 = H(f(s), 1) = k(f(s)) = k \circ f \quad (9)$$

While we define $\alpha(t)$ as $H(x_0, t) = H \circ c$. $(† † †)$

Notice that $H(x_0, 0) = h(x_0) = y_0$ and $H(x_0, 1) = k(x_0) = y_1$. So in fact $\alpha$ is a path from $y_0$ to $y_1$. 

Lemma 58.4 Continued

Since $H$ is a homotopy between $h$ and $k$, then

$$(†) \quad H \circ f_0 = H(f(s), 0) = h(f(s)) = h \circ f$$

$$(††) \quad H \circ f_1 = H(f(s), 1) = k(f(s)) = k \circ f$$

While we define $α(t)$ as $H(x_0, t) = H \circ c$. ($† † †$)

Notice that $H(x_0, 0) = h(x_0) = y_0$ and $H(x_0, 1) = k(x_0) = y_1$. So in fact $α$ is a path from $y_0$ to $y_1$.

Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the following paths in $I \times I$ which run along the four edges of $I \times I$:

$$β_0(s) = (s, 0) \quad β_1(s) = (s, 1)$$

$$γ_0(t) = (0, t) \quad γ_1(t) = (1, t)$$

Then $F \circ β_0 = F(s, 0) = (f(s), 0)$ and $F \circ β_1 = F(s, 1) = (f(s), 1)$, while $F \circ γ_0 = F(0, t) = (f(0), t) = (x_0, t) = c(t)$ and $F \circ γ_1 = F(1, t) = (f(1), t) = (x_0, t) = c(t)$. 
Since $H$ is a homotopy between $h$ and $k$, then

\begin{align*}
(†) \quad H \circ f_0 &= H(f(s), 0) = h(f(s)) = h \circ f \\
(††) \quad H \circ f_1 &= H(f(s), 1) = k(f(s)) = k \circ f
\end{align*}

While we define $\alpha(t)$ as $H(x_0, t) = H \circ c$. († † † †)

Notice that $H(x_0, 0) = h(x_0) = y_0$ and $H(x_0, 1) = k(x_0) = y_1$. So in fact $\alpha$ is a path from $y_0$ to $y_1$.

Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the following paths in $I \times I$ which run along the four edges of $I \times I$:

\begin{align*}
\beta_0(s) &= (s, 0) \quad \beta_1(s) = (s, 1) \\
\gamma_0(t) &= (0, t) \quad \gamma_1(t) = (1, t)
\end{align*}

Then $F \circ \beta_0 = F(s, 0) = (f(s), 0)$ and $F \circ \beta_1 = F(s, 1) = (f(s), 1)$, while $F \circ \gamma_0 = F(0, t) = (f(0), t) = (x_0, t) = c(t)$ and $F \circ \gamma_1 = F(1, t) = (f(1), t) = (x_0, t) = c(t)$. 
Lemma 58.4 Continued

Now paths $\beta_0 \ast \gamma_1$ and $\gamma_0 \ast \beta_1$ are paths in $I \times I$ (along the boundary) from $(0, 0)$ to $(1, 1)$. Since $I \times I$ is convex (See Example 51.1), there is a path homotopy $G$ between them (say from $\beta_0 \ast \gamma_1$ to $\gamma_0 \ast \beta_1$).

Then for $F \circ G : (X \times I) \times I$ we have

$$(F \circ G)(\beta_0 \ast \gamma_1, 0) = F(\beta_0 \ast \gamma_1),$$
$$(F \circ G)(\beta_0 \ast \gamma_1, 0) = F(\beta_0 \ast \gamma_1) = F(\beta_0) \ast F(\gamma_1) = (f(s), 0) \ast c(t) = f_0 \ast c,$$

and

$$(F \circ G)(\beta_0 \ast \gamma_1, 1) = F(\gamma_0 \ast \beta_1) = F(\gamma_0) \ast F(\beta_1) = c(t) \ast (f(s), 1) = c \ast f_1.$$
Lemma 58.4 Continued

Now paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ (along the boundary) from $(0, 0)$ to $(1, 1)$. Since $I \times I$ is convex (See Example 51.1), there is a path homotopy $G$ between them (say from $\beta_0 * \gamma_1$ to $\gamma_0 * \beta_1$).

Then for $F \circ G : (X \times I) \times I$ we have $(F \circ G)(\beta_0 * \gamma_1, 0) = F(\beta_0 * \gamma_1)$, $(F \circ G)(\beta_0 * \gamma_1, 0) = F(\beta_0 * \gamma_1) = F(\beta_0) * F(\gamma_1) = (f(s), 0) * c(t) = f_0 * c$, and

$(F \circ G)(\beta_0 * \gamma_1, 1) = F(\gamma_0 * \beta_1) = F(\gamma_0) * F(\beta_1) = c(t) * (f(s), 1) = c * f_1$.

So $F \circ G$ is a path homotopy in $X \times I$ from $f_0 * c$ to $c * f_1$. Now consider $H \circ (F \circ G)$. Recall $H : X \times I \rightarrow Y$ and notice that $f_0 * c, c * f_1 \in X \times I$ (See Figure 58.3)
Lemma 58.4 Continued

Now paths $\beta_0 \ast \gamma_1$ and $\gamma_0 \ast \beta_1$ are paths in $I \times I$ (along the boundary) from $(0, 0)$ to $(1, 1)$. Since $I \times I$ is convex (See Example 51.1), there is a path homotopy $G$ between them (say from $\beta_0 \ast \gamma_1$ to $\gamma_0 \ast \beta_1$).

Then for $F \circ G : (X \times I) \times I$ we have

$$(F \circ G)(\beta_0 \ast \gamma_1, 0) = F(\beta_0 \ast \gamma_1),$$

$$(F \circ G)(\beta_0 \ast \gamma_1, 0) = F(\beta_0 \ast \gamma_1) = F(\beta_0) \ast F(\gamma_1) = (f(s), 0) \ast c(t) = f_0 \ast c,$$

and

$$(F \circ G)(\beta_0 \ast \gamma_1, 1) = F(\gamma_0 \ast \beta_1) = F(\gamma_0) \ast F(\beta_1) = c(t) \ast (f(s), 1) = c \ast f_1.$$

So $F \circ G$ is a path homotopy in $X \times I$ from $f_0 \ast c$ to $c \ast f_1$. Now consider $H \circ (F \circ G)$. Recall $H : X \times I \rightarrow Y$ and notice that $f_0 \ast c, c \ast f_1 \in X \times I$ (See Figure 58.3)
Lemma 58.4 Continued

Apply $H \circ (F \circ G)$ to $(\beta_0 \ast \gamma_1, 0)$ gives

$$H \circ (F \circ G)(\beta_0 \ast \gamma_1, 0) = H(f_0 \ast c)$$
$$= H(f_0) \ast H(c) = (H \circ f_0) \ast (H \circ c) \quad (11)$$
$$= (h \circ f) \ast \alpha \text{ by (†) and († † †)}$$

and

$$H \circ (F \circ G)(\beta_0 \ast \gamma_1, 1) = H(c \ast f_1)$$
$$= H(c) \ast H(f_1) = (H \circ c) \ast (H \circ f_1) \quad (12)$$
$$= \alpha \ast (k \circ f) \text{ by (††) and († † †)}$$

So $H \circ (F \circ G)$ is a path homotopy between $(h \circ f) \ast \alpha$ and $\alpha \ast (k \circ f)$. That is,

$$[\alpha \ast (k \circ f)] = [(h \circ f) \ast \alpha] \text{ or } [\alpha] \ast [k \circ f] = [h \circ f] \ast [\alpha] \quad (13)$$

This is the equation (*) that we needed to verify.
Lemma 58.4 Continued

Apply $H \circ (F \circ G)$ to $(\beta_0 \ast \gamma_1, 0)$ gives

$$H \circ (F \circ G)(\beta_0 \ast \gamma_1, 0) = H(f_0 \ast c)$$
$$= H(f_0) \ast H(c) = (H \circ f_0) \ast (H \circ c)$$  \hspace{1cm} (11)
$$= (h \circ f) \ast \alpha \text{ by } (\dagger) \text{ and } (\dagger \dagger \dagger)$$

and

$$H \circ (F \circ G)(\beta_0 \ast \gamma_1, 1) = H(c \ast f_1)$$
$$= H(c) \ast H(f_1) = (H \circ c) \ast (H \circ f_1)$$  \hspace{1cm} (12)
$$= \alpha \ast (k \circ f) \text{ by } (\dagger \dagger \dagger) \text{ and } (\dagger \dagger \dagger)$$

So $H \circ (F \circ G)$ is a path homotopy between $(h \circ f) \ast \alpha$ and $\alpha \ast (k \circ f)$. That is,

$$[\alpha \ast (k \circ f)] = [(h \circ f) \ast \alpha] \text{ or }$$
$$[\alpha] \ast [k \circ f] = [h \circ f] \ast [\alpha]$$  \hspace{1cm} (13)

This is the equation (*) that we needed to verify.
Corollary 58.5. Let $h, k : X \to Y$ be homotopic continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If induced homomorphism $h_*$ is injective (one to one), surjective (onto), or trivial then so is the induced homomorphism $k_*$.

Proof. By Lemma 58.4, there exists a path from $y_0$ to $y_1$, such that $k_* = \hat{\alpha} \circ h_*$. By Theorem 52.1, $\hat{\alpha}$ is a group isomorphism and so is one to one and onto.
Corollary 58.5. Let $h, k : X \to Y$ be homotopic continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If induced homomorphism $h_\ast$ is injective (one to one), surjective (onto), or trivial then so is the induced homomorphism $k_\ast$.

Proof. By Lemma 58.4, there exists a path from $y_0$ to $y_1$, such that $k_\ast = \hat{\alpha} \circ h_\ast$. By Theorem 52.1, $\hat{\alpha}$ is a group isomorphism and so is one to one and onto. So if $h_\ast$ is one to one/onto/trivial, then so is $k_\ast$. 

\qed
Corollary 58.5. Let $h, k : X \rightarrow Y$ be homotopic continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If induced homomorphism $h_*$ is injective (one to one), surjective (onto), or trivial then so is the induced homomorphism $k_*$.  

Proof. By Lemma 58.4, there exists a path from $y_0$ to $y_1$, such that $k_* = \hat{\alpha} \circ h_*$. By Theorem 52.1, $\hat{\alpha}$ is a group isomorphism and so is one to one and onto. So if $h_*$ is one to one/onto/trivial, then so is $k_*$.  \[\square\]
Corollary 58.6. Let $h : X \to Y$. If $h$ is nulhomotopic, then $h_*$ is the trivial homomorphism.

**Proof.** Let $h$ be homotopic to a constant with $h(x_0) = y_0$. Define $i(x) = y_0$ for all $x \in X$, then by Theorem 58.4, we have that $i_* = \hat{\alpha} \circ h_*$ where $\alpha$ is a path from $y_0$ to $y_0$. The constant map $i$ induces the trivial homomorphism, so $\hat{\alpha} \circ h_*$ must be the trivial homomorphism and $h_*$ must be trivial.
Corollary 58.6. Let \( h : X \to Y \). If \( h \) is nulhomotopic, then \( h_* \) is the trivial homomorphism.

**Proof.** Let \( h \) be homotopic to a constant with \( h(x_0) = y_0 \). Define \( i(x) = y_0 \) for all \( x \in X \), then by Theorem 58.4, we have that \( i_* = \hat{\alpha} \circ h_* \) where \( \alpha \) is a path from \( y_0 \) to \( y_0 \). The constant map \( i \) induces the trivial homomorphism, so \( \hat{\alpha} \circ h_* \) must be the trivial homomorphism and \( h_* \) must be trivial.
Theorem 58.7

Theorem 58.7. Let $f : X \to Y$ be continuous. Let $f(x_0) = y_0$. If $f$ is a homotopy equivalence, then the induced homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Proof. Every homotopy is invertible, so let $g : Y \to X$ be a homotopy inverse for $f$. Consider the maps

$$\begin{align*}
(X, x_0) &\xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)
\end{align*}$$

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. The corresponding induced homomorphisms (which depend on the base point and this is indicated when necessary with subscript):

$$\begin{align*}
\pi_1(X, x_0) &\xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1)
\end{align*}$$
Theorem 58.7

Theorem 58.7. Let $f : X \to Y$ be continuous. Let $f(x_0) = y_0$. If $f$ is a homotopy equivalence, then the induced homomorphism

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(14)

is an isomorphism.

Proof. Every homotopy is invertible, so let $g : Y \to X$ be a homotopy inverse for $f$. Consider the maps

$$
(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)
$$

(15)

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. The corresponding induced homomorphisms (which depend on the base point and this is indicated when necessary with subscript):

$$
\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1)
$$

(16)
Theorem 58.7

Since $g$ is a homotopy inverse then

$$g \circ f : (X, x_0) \rightarrow (X, x_1)$$

(17)

is homotopic to the identity map, so by Lemma 58.4 there is a path $\alpha$ in $X$ from $x_0$ to $x_1$ such that

$$(g \circ f)_* = \hat{\alpha} \circ (\iota_x)_* = \hat{\alpha}$$

(18)

since the identity map induces the identity group homomorphism. Now $\hat{\alpha}$ is a group isomorphism by Theorem 52.1, so

$$\hat{\alpha} = (g \circ f)_* = g_* \circ (f_{x_0})_*$$

is an isomorphism.
Theorem 58.7

Since $g$ is a homotopy inverse then

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since the identity map induces the identity group homomorphism. Now $\hat{\alpha}$ is a group isomorphism by Theorem 52.1, so

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is an isomorphism.

Similarly, because $f \circ g$ is homotopic to the identity map $\iota_*$, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism.
Theorem 58.7

Since $g$ is a homotopy inverse then

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is homotopic to the identity map, so by Lemma 58.4 there is a path $\alpha$ in $X$ from $x_0$ to $x_1$ such that

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since the identity map induces the identity group homomorphism. Now $\hat{\alpha}$ is a group isomorphism by Theorem 52.1, so

$$\hat{\alpha} = (g \circ f)_* = g_* \circ (f_{x_0})_*$$

is an isomorphism.

Similarly, because $f \circ g$ is homotopic to the identity map $\iota_*$, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism.
The fact that $g_* \circ (f_{x_0})_*$ is an isomorphism implies that $g_*$ is surjective (onto). The fact that $(f_{x_1})_* \circ g_*$ is an isomorphism implies that $g_*$ is injective (one to one). So $g_*$ is an isomorphism and hence so is $(g_*)^{-1}$. Since $g_* \circ (f_{x_0})_* = \hat{\alpha}$ from above, then $(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$ is an isomorphism as claimed.
Theorem 58.7

The fact that $g_* \circ (f_{x_0})_*$ is an isomorphism implies that $g_*$ is surjective (onto). The fact that $(f_{x_1})_* \circ g_*$ is an isomorphism implies that $g_*$ is injective (one to one). So $g_*$ is an isomorphism and hence so is $(g_*)^{-1}$. Since $g_* \circ (f_{x_0})_* = \hat{\alpha}$ from above, then $(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$ is an isomorphism as claimed.