Chapter 9. The Fundamental Group
Section 59. The Fundamental Group of $S^n$—Proofs of Theorems
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Theorem 59.1. Suppose \( X = U \cup V \) where \( U \) and \( V \) are open sets of \( X \). Suppose that \( U \cap V \) is path connected and that \( x_0 \in U \cap V \). Let \( i \) and \( j \) be the inclusion mappings of \( U \) and \( V \), respectively, into \( X \). Then the images of the induced homomorphisms

\[
i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)
\]

(1) generate the group \( \pi_1(X, x_0) \).

Proof. Recall that if group \( G \) is generated by elements \( a_i \in G \) where \( i \in I \), then the elements of \( G \) are all finite products of integer powers of the \( a_i \) (Fraleigh’s Theorem 7.6). So the claim of this theorem is that any loop \( f \) in \( X \) based at \( x_0 \) is path homotopic to a product of the form \((g_1 \ast (g_2 \ast (\ldots \ast g_n)))\) where each \( g_i \) is a loop in \( X \) based at \( x_0 \) that lies either in \( U \) or \( V \).
Theorem 59.1

**Theorem 59.1.** Suppose $X = U \cup V$ where $U$ and $V$ are open sets of $X$. Suppose that $U \cap V$ is path connected and that $x_0 \in U \cap V$. Let $i$ and $j$ be the inclusion mappings of $U$ and $V$, respectively, into $X$. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and } j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

(1)

generate the group $\pi_1(X, x_0)$.

**Proof.** Recall that if group $G$ is generated by elements $a_i \in G$ where $i \in I$, then the elements of $G$ are all finite products of integer powers of the $a_i$ (Fraleigh’s Theorem 7.6). So the claim of this theorem is that any loop $f$ in $X$ based at $x_0$ is path homotopic to a product of the form $(g_1 * (g_2 * (\ldots * g_n)))$ where each $g_i$ is a loop in $X$ based at $x_0$ that lies either in $U$ or $V$. 


STEP 1 Choose a subdivision $0 = b_0 < b_1 < \ldots < b_m = 1$ of $[0, 1]$ such that for each $i$, the set $f([b_{i-1}, b_i])$ is contained in either $U$ or $V$ (which can be done since path $f$ in $X$ is compact) (Munkres cites the Lebesgue Number Lemma [],). If $f(b_i) \in U \cap V$ for all $i$, we stop. If not, let $i$ be an index such that $f(b_i) \notin U \cap V$. 
Theorem 59.1

**STEP 1** Choose a subdivision $0 = b_0 < b_1 < \ldots < b_m = 1$ of $[0, 1]$ such that for each $i$, the set $f([b_{i-1}, b_i])$ is contained in either $U$ or $V$ (which can be done since path $f$ in $X$ is compact) (Munkres cites the Lebesgue Number Lemma [ ],) If $f(b_i) \in U \cap V$ for all $i$, we stop. If not, let $i$ be an index such that $f(b_i) \notin U \cap V$. For this index value, each of the sets $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ lies either in $U$ or in $V$. If $f(b_i) \in U$ then both of these sets must lie in $U$; if $f(b_i) \in V$ then both of these sets must lie in $V$. 


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$$0 = b_0 < b_1 < ... < b_{i-1} < b_i < b_{i+1} < ... < b_m = 1$$ (2)
**Theorem 59.1**

**STEP 1** Choose a subdivision $0 = b_0 < b_1 < ... < b_m = 1$ of $[0, 1]$ such that for each $i$, the set $f([b_{i-1}, b_i])$ is contained in either $U$ or $V$ (which can be done since path $f$ in $X$ is compact) (Munkres cites the Lebesgue Number Lemma [ ],) If $f(b_i) \in U \cap V$ for all $i$, we stop. If not, let $i$ be an index such that $f(b_i) \notin U \cap V$. For this index value, each of the sets $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ lies either in $U$ or in $V$. If $f(b_i) \in U$ then both of these sets must lie in $U$; if $f(b_i) \in V$ then both of these sets must lie in $V$. In either case, delete $b_i$ from the partition, producing the new partition

$$0 = b_0 < b_1 < ... < b_{i-1} < b_i < b_{i+1} < ... < b_m = 1$$

(2)

Perform this process over each index value and the process yields a partition $0 = a_0 < a_1 < ... < a_n = 1$ of $[0, 1]$ such that $f(a_i) \in U \cap V$ for all $i$ and $f([a_{i-1}, a_i])$ is contained either in $U$ or in $V$ for all $i$. 
**Theorem 59.1**

**STEP 1** Choose a subdivision \(0 = b_0 < b_1 < \ldots < b_m = 1\) of \([0, 1]\) such that for each \(i\), the set \(f([b_{i-1}, b_i])\) is contained in either \(U\) or \(V\) (which can be done since path \(f\) in \(X\) is compact) (Munkres cites the Lebesgue Number Lemma \([\text{ ]}\))

If \(f(b_i) \in U \cap V\) for all \(i\), we stop. If not, let \(i\) be an index such that \(f(b_i) \notin U \cap V\). For this index value, each of the sets \(f([b_{i-1}, b_i])\) and \(f([b_i, b_{i+1}])\) lies either in \(U\) or in \(V\). If \(f(b_i) \in U\) then both of these sets must lie in \(U\); if \(f(b_i) \in V\) then both of these sets must lie in \(V\). In either case, delete \(b_i\) from the partition, producing the new partition

\[0 = b_0 < b_1 < \ldots < b_{i-1} < b_i < b_{i+1} < \ldots < b_m = 1\]  

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Perform this process over each index value and the process yields a partition \(0 = a_0 < a_1 < \ldots < a_n = 1\) of \([0, 1]\) such that \(f(a_i) \in U \cap V\) for all \(i\) and \(f([a_{i-1}, a_i])\) is contained either in \(U\) or in \(V\) for all \(i\).
Theorem 59.1

STEP 2 Given \( f \), let \( 0 = a_0 < a_1 < \ldots < a_n = 1 \) be a partition of the sort constructed in STEP 1. Define \( f_i \) to be the path in \( X \) that equals the positive linear map of \([0, 1]\) onto \([a_{i-1}, a_i]\) followed by \( f \); so \( f_i : [0, 1] \to f|_{[a_{i-1}, a_i]} \).

So \( f_i \) is a path that lies either in \( U \) or in \( V \), and by Theorem 51.3, \( [f] = [f_1] \ast [f_2] \ast \ldots \ast [f_n] \).
Theorem 59.1

**STEP 2** Given \( f \), let \( 0 = a_0 < a_1 < \ldots < a_n = 1 \) be a partition of the sort constructed in STEP 1. Define \( f_i \) to be the path in \( X \) that equals the positive linear map of \([0, 1]\) onto \([a_{i-1}, a_i]\) followed by \( f \); so \( f_i : [0, 1] \to f|_{[a_{i-1}, a_i]} \).

So \( f_i \) is a path that lies either in \( U \) or in \( V \), and by Theorem 51.3, \( [f] = [f_1] \ast [f_2] \ast \ldots \ast [f_n] \).

For each index \( i \), choose a path \( \alpha_i \) in \( U \cap V \) from \( x_0 \) to \( f(a_i) \) (which can be done since \( U \cap V \) is path connected). Since \( f(a_0) = f(a_n) = x_0 \), we can choose \( \alpha_0 \) and \( \alpha_n \) to be the constant path at \( x_0 \).
STEP 2 Given $f$, let $0 = a_0 < a_1 < ... < a_n = 1$ be a partition of the sort constructed in STEP 1. Define $f_i$ to be the path in $X$ that equals the positive linear map of $[0, 1]$ onto $[a_{i-1}, a_i]$ followed by $f$; so $f_i : [0, 1] \rightarrow f|_{[a_{i-1}, a_i]}$.

So $f_i$ is a path that lies either in $U$ or in $V$, and by Theorem 51.3, $[f] = [f_1] * [f_2] * ... * [f_n]$.

For each index $i$, choose a path $\alpha_i$ in $U \cap V$ from $x_0$ to $f(a_i)$ (which can be done since $U \cap V$ is path connected). Since $f(a_0) = f(a_n) = x_0$, we can choose $\alpha_0$ and $\alpha_n$ to be the constant path at $x_0$. 
Theorem 59.1

Now set \( f_i = (\alpha_{i-1} * f_i) * \bar{\alpha}_i \) for each \( i \). Then \( g_i \) is a loop in \( X \) based at \( x_0 \) whose image lies either in \( U \) or in \( V \). Now

\[
[g_1] * [g_2] * [g_3] * \ldots * [g_n] = [\alpha_0 * f_1 * \bar{\alpha}_1] * [\alpha_1 * f_2 * \bar{\alpha}_2] * [\alpha_2 * f_3 * \bar{\alpha}_3] * \ldots * [\alpha_{n-1} * f_n * \bar{\alpha}_n] \\
= [\alpha_0] * [f_1] * [\bar{\alpha}_1] * [\alpha_1] * [f_2] * [\bar{\alpha}_2] * [\alpha_2] * [f_3] * [\bar{\alpha}_3] * \ldots * [\alpha_{n-1}] * [f_n] * [\bar{\alpha}_n] \text{ by definition of } [\alpha_{i-1} * f_i * \bar{\alpha}_i] \\
= [f_1] * [f_2] * \ldots * [f_n] \\
= [f] \tag{3}
\]

So arbitrary path \( f \) is path homotopic to a product of loops \( g_i \) where each \( g_i \) is a loop in \( X \) based at \( x_0 \) whose image lies either in \( U \) or in \( V \). That is, either \([g_i] \in \pi_1(U, x_0)\) or \([g_i] \in \pi_1(V, x_0)\) for all \( i \).
Theorem 59.1

Now set $f_i = (\alpha_{i-1} * f_i) * \bar{\alpha}_i$ for each $i$. Then $g_i$ is a loop in $X$ based at $x_0$ whose image lies either in $U$ or in $V$. Now

$$[g_1] * [g_2] * [g_3] * ... * [g_n] = [\alpha_0 * f_1 * \bar{\alpha}_1] * [\alpha_1 * f_2 * \bar{\alpha}_2] * [\alpha_2 * f_3 * \bar{\alpha}_3] *$$

$$... * [\alpha_{n-1} * f_n * \bar{\alpha}_n]$$

$$= [\alpha_0] * [f_1] * [\bar{\alpha}_1] * [\alpha_1] * [f_2] * [\bar{\alpha}_2] * [\alpha_2] * [f_3] *$$

$$[\bar{\alpha}_3] * ... * [\alpha_{n-1}] * [f_n] * [\bar{\alpha}_n] \text{ by definition}$$

$$of [\alpha_{i-1} * f_i * \bar{\alpha}_i]$$

$$= [f_1] * [f_2] * ... * [f_n]$$

$$= [f]$$

(3)

So arbitrary path $f$ is path homotopic to a product of loops $g_i$ where each $g_i$ is a loop in $X$ based at $x_0$ whose image lies either in $U$ or in $V$. That is, either $[g_i] \in \pi_1(U, x_0)$ or $[g_i] \in \pi_1(V, x_0)$ for all $i$. □
Corollary 59.2. Suppose \( X = U \cup V \) where \( U \) and \( V \) are open sets of \( X \). Suppose \( U \cap V \) is nonempty and path connected. If \( U \) and \( V \) are simply connected then \( X \) is simply connected.

Proof. By the definition of simply connected, we know that \( U \) and \( V \) are path connected and \( \pi_1(U, x_0) \cong \pi_1(V, x_0) \cong \{e\} \) for some \( x_0 \in U \cap V \). The hypothesis of Theorem 59.1 are satisfied and the images of \( i_* \) and \( j_* \) as given in Theorem 59.1 consist of the identity of \( \pi_1(X, x_0) \cong \{e\} \). Since \( U \) and \( V \) are path connected and \( U \cap V \) is nonempty, then \( X = U \cup V \) is path connected. So by definition, \( X \) is simply connected. \( \square \)
Corollary 59.2. Suppose $X = U \cup V$ where $U$ and $V$ are open sets of $X$. Suppose $U \cap V$ is nonempty and path connected. If $U$ and $V$ are simply connected then $X$ is simply connected.

**Proof.** By the definition of simply connected, we know that $U$ and $V$ are path connected and $\pi_1(U, x_0) \cong \pi_1(V, x_0) \cong \{e\}$ for some $x_0 \in U \cap V$. The hypothesis of Theorem 59.1 are satisfied and the images of $i_*$ and $j_*$ as given in Theorem 59.1 consist of the identity of $\pi_1(X, x_0) \cong \{e\}$. Since $U$ and $V$ are path connected and $U \cap V$ is nonempty, then $X = U \cup V$ is path connected. So by definition, $X$ is simply connected. □
Theorem 59.3

**Theorem 59.3.** If $n \geq 2$, the $n$-sphere is simply connected.

**Proof.** Let $\bar{p} = (0, 0, 0, 1) \in \mathbb{R}^{n+1}$ and $\bar{q} = (0, 0, ..., -1)$ be the "north pole" and the "south pole" of $S^n$, respectively, where $S^n$ is considered as embedded in $\mathbb{R}^{n+1}$ as

$$S^n = \{(x_1, x_2, ..., x_{n+1}) | x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}. \quad (4)$$

**STEP 1** Define $f_i(S^n - \{\bar{p}\}) \rightarrow \mathbb{R}^n$ by the equation

$$f(\bar{x}) = f(x_1, ..., x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n). \quad (5)$$
Theorem 59.3. If $n \geq 2$, the $n$-sphere is simply connected.

Proof. Let $\vec{p} = (0, 0, 0, 1) \in \mathbb{R}^{n+1}$ and $\vec{q} = (0, 0, \ldots, -1)$ be the "north pole" and the "south pole" of $S^n$, respectively, where $S^n$ is considered as embedded in $\mathbb{R}^{n+1}$ as

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**STEP 1** Define $f_i(S^n - \{\vec{p}\}) \rightarrow \mathbb{R}^n$ by the equation

$$f(\vec{x}) = f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n). \quad (5)$$

The map $f$ is called the stereographic projection. (If we take the line in $\mathbb{R}^{n+1}$ through $\vec{p}$ and $\vec{x} \in S^n - \{\vec{p}\}$ then this line intersects the $n$-plane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ at the point $f(\vec{x}) \times \{0\}$. This projection is used in complex analysis to map $S^2$ to the extended complex plane.)
Theorem 59.3

**Theorem 59.3.** If $n \geq 2$, the $n$-sphere is simply connected.

**Proof.** Let $\vec{p} = (0, 0, 0, 1) \in \mathbb{R}^{n+1}$ and $\vec{q} = (0, 0, ..., -1)$ be the "north pole" and the "south pole" of $S^n$, respectively, where $S^n$ is considered as embedded in $\mathbb{R}^{n+1}$ as

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Theorem 59.3

Consider the map \( g : \mathbb{R}^n \to (S^n - \{\vec{p}\}) \) given by

\[
g(\vec{y}) = g(y_1, \ldots, y_n) = (t(y) \cdot y_1, \ldots, t(y) \cdot y_n, 1 - t(y))
\]

where \( t(y) = \frac{2}{(1 + \|\vec{y}\|^2)} \). Then \( g \) is a left and right inverse of \( f \). So \( f \) is a bijection, \( f \) is continuous on \( S^n - \{\vec{p}\} \), and \( f^{-1} = g \) is continuous on \( \mathbb{R}^n \). So \( f \) is a homeomorphism between \( S^n - \{\vec{p}\} \) and \( \mathbb{R}^n \).
Consider the map \( g : \mathbb{R}^n \to (S^n - \{ \vec{p} \}) \) given by

\[
g(\vec{y}) = g(y_1, ..., y_n) = (t(y) \cdot y_1, ..., t(y) \cdot y_n, 1 - t(y)) \tag{6}
\]

where \( t(y) = \frac{2}{(1 + \| \vec{y} \|^2)} \). Then \( g \) is a left and right inverse of \( f \). So \( f \) is a bijection, \( f \) is continuous on \( S^n - \{ \vec{p} \} \), and \( f^{-1} = g \) is continuous on \( \mathbb{R}^n \). So \( f \) is a homeomorphism between \( S^n - \{ \vec{p} \} \) and \( \mathbb{R}^n \).

Note that the reflection map \((x_1, ..., x_n, x_{n+1}) \to (x_1, ..., x_n, -x_{n+1})\) defines a homeomorphism of \( S^n - \{ \vec{p} \} \) with \( S^n - \{ \vec{q} \} \), so \( S^n - \{ \vec{q} \} \) is also homeomorphic to \( \mathbb{R}^n \).
Consider the map $g : \mathbb{R}^n \rightarrow (S^n - \{\vec{p}\})$ given by

$$g(\vec{y}) = g(y_1, ..., y_n) = (t(y) \cdot y_1, ..., t(y) \cdot y_n, 1 - t(y))$$  \hspace{1cm} (6)

where $t(y) = \frac{2}{(1 + \|\vec{y}\|^2)}$. Then $g$ is a left and right inverse of $f$. So $f$ is a bijection, $f$ is continuous on $S^n - \{\vec{p}\}$, and $f^{-1} = g$ is continuous on $\mathbb{R}^n$. So $f$ is a homeomorphism between $S^n - \{\vec{p}\}$ and $\mathbb{R}^n$.

Note that the reflection map $(x_1, ..., x_n, x_{n+1}) \rightarrow (x_1, ..., x_n, -x_{n+1})$ defines a homeomorphism of $S^n - \{\vec{p}\}$ with $S^n - \{\vec{q}\}$, so $S^n - \{\vec{q}\}$ is also homeomorphic to $\mathbb{R}^n$. 
**Theorem 59.3**

**STEP 2** Let $U = S^n - \{\vec{p}\}$ and $V = S^n - \{\vec{q}\}$. Then $U$ and $V$ are open sets in $S^n$.

First, for $n \geq 1$ the sphere $S^n$ is path connected since $U$ and $V$ are path connected (they are homeomorphic to $\mathbb{R}^n$ by STEP 1) and have the point $(1, 0, ..., 0)$ of $S^n$ in common [for example].
Theorem 59.3

STEP 2 Let $U = S^n - \{ \bar{p} \}$ and $V = S^n - \{ \bar{q} \}$. Then $U$ and $V$ are open sets in $S^n$.

First, for $n \geq 1$ the sphere $S^n$ is path connected since $U$ and $V$ are path connected (they are homeomorphic to $\mathbb{R}^n$ by STEP 1) and have the point $(1, 0, ..., 0)$ of $S^n$ in common [for example].

The space $U$ and $V$ are also simply connected, since they are homeomorphic to $\mathbb{R}^n$. $U \cap V = S^n \{ \bar{p}, \bar{q} \}$, which is homeomorphic under stereographic projection to $\mathbb{R}^n \{ (0, 0) \}$ (since stereographic projection maps $\bar{q}$ to $(0, 0)$).
STEP 2 Let $U = S^n - \{\vec{p}\}$ and $V = S^n - \{\vec{q}\}$. Then $U$ and $V$ are open sets in $S^n$.

First, for $n \geq 1$ the sphere $S^n$ is path connected since $U$ and $V$ are path connected (they are homeomorphic to $\mathbb{R}^n$ by STEP 1) and have the point $(1, 0, ..., 0)$ of $S^n$ in common [for example].

The space $U$ and $V$ are also simply connected, since they are homeomorphic to $\mathbb{R}^n$. $U \cap V = S^n \setminus \{\vec{p}, \vec{q}\}$, which is homeomorphic under stereographic projection to $\mathbb{R}^n \setminus \{(0, 0)\}$ (since stereographic projection maps $\vec{q}$ to $(0, 0)$). Since $n \geq 2$, $\mathbb{R}^n \setminus \{(0, 0)\}$ is path connected because every point of $\mathbb{R}^n \setminus \{(0, 0)\}$ can be joined to a point of $S^{n-1}$ by a straight-line path and $S^{n-1}$ is path connected. So the hypotheses of Corollary 59.2 hold and $S^n$ is simply connected.
**Theorem 59.3**

**STEP 2** Let $U = S^n - \{\vec{p}\}$ and $V = S^n - \{\vec{q}\}$. Then $U$ and $V$ are open sets in $S^n$.

First, for $n \geq 1$ the sphere $S^n$ is path connected since $U$ and $V$ are path connected (they are homeomorphic to $\mathbb{R}^n$ by STEP 1) and have the point $(1, 0, ..., 0)$ of $S^n$ in common [for example].

The space $U$ and $V$ are also simply connected, since they are homeomorphic to $\mathbb{R}^n$. $U \cap V = S^n \backslash \{\vec{p}, \vec{q}\}$, which is homeomorphic under stereographic projection to $\mathbb{R}^n \backslash \{(0, 0)\}$ (since stereographic projection maps $\vec{q}$ to $(0, 0)$). Since $n \geq 2$, $\mathbb{R}^n \backslash \{(0, 0)\}$ is path connected because every point of $\mathbb{R}^n \backslash \{(0, 0)\}$ can be joined to a point of $S^{n-1}$ by a straight-line path and $S^{n-1}$ is path connected. So the hypotheses of Corollary 59.2 hold and $S^n$ is simply connected.