Section 14. The Order Topology

Note. Munkres defines an order relation (which he refers to in this section as a “simple order”), denoted “<,” on a set \( A \) as a relation (see page 21) satisfying:

1. **Comparability:** For every \( x, y \in A \) for which \( x \neq y \), either \( x < y \) or \( y < x \).
2. **Nonreflexivity:** For no \( x \in A \) does the relation \( x < x \) hold.
3. **Transitivity:** If \( x < y \) and \( y < z \) then \( x < z \).

In this section, we use a simple order relation on a set to define a topology on the set.

**Definition.** Let \( X \) be a set. A *basis* for a topology on \( X \) is a collection \( \mathcal{B} \) of subsets of \( X \) (called *basis elements*) such that

1. For each \( x \in X \), there is at least one basis element \( B \in \mathcal{B} \) such that \( x \in B \).
2. If \( x \in B_1 \cap B_2 \) where \( B_1, B_2 \in \mathcal{B} \) then there is \( B_3 \in \mathcal{B} \) such that \( x \in B_3 \) and \( B_3 \subset B_2 \cap B_2 \).

The topology \( \mathcal{T} \) *generated by* \( \mathcal{B} \) is defined as: A subset \( U \subset X \) is in \( \mathcal{T} \) if for each \( x \in U \) there is \( B \in \mathcal{B} \) such that \( x \in B \) and \( B \subset U \). (Therefore each basis element is in \( \mathcal{T} \).)
**Definition.** Let \( X \) be a set with a simple order relation \(<\). The following sets are *intervals* in \( X \):

\[
(a, b) = \{ x \in X \mid a < x < b \} \quad \text{(open intervals)}
\]

\[
(a, b] = \{ x \in X \mid a < x \leq b \} \quad \text{(half-open intervals)}
\]

\[
[a, b) = \{ x \in X \mid a \leq x < b \} \quad \text{(half-open intervals)}
\]

\[
[a, b] = \{ x \in X \mid a \leq x \leq b \} \quad \text{(closed intervals)}.
\]

**Definition.** Let \( X \) be a set with a simple order relation and assume \( X \) has more than one element. Let \( \mathcal{B} \) be the collection of all sets of the following types:

1. All open intervals \((a, b)\) in \( X \).
2. All intervals of the form \([a_0, b)\) where \( a_0 \) is the least element (if one exists) of \( X \).
3. All intervals of the form \((a, b_0]\) where \( b_0 \) is the greatest element (if one exists) of \( X \).

The collection \( \mathcal{B} \) is a basis for a topology on \( X \) called the *order topology*.

**Note.** Of course we must verify that \( \mathcal{B} \) really is a basis for a topology.

**Theorem 14.A.** Let \( X \) be a set with a simple order relation and let \( \mathcal{B} \) consist of all open intervals \((a, b)\), all intervals \([a_0, b)\), and all intervals \((a, b_0]\), where \( a_0 \) is the least element of \( X \) and \( b_0 \) is the greatest element of \( X \) (if such exist). Then \( \mathcal{B} \) is a basis for a topology on \( X \).
**Example 1.** The standard topology on $\mathbb{R}$ is the order topology based on the usual “less than” order on $\mathbb{R}$.

**Example 2.** We can put a simple order relation on $\mathbb{R}^2$ as follows: $(a, b) < (c, d)$ if either (1) $a < c$, or (2) $a = c$ and $b < d$. This is often called the lexicographic ordering (see my Complex Analysis 1 [MATH 5510] notes for a mention on the lexicographic ordering applied to $\mathbb{C}$: [http://faculty.etsu.edu/gardnerr/5510/Ordering-C.pdf](http://faculty.etsu.edu/gardnerr/5510/Ordering-C.pdf)) or, as Munkres calls it, the dictionary order. These two types of open intervals under this simple order relation are then as follows:

![Diagram](image-url)

Notice that this can easily be generalized to $\mathbb{R}^n$. 
**Example 4.** Let $X = \{1, 2\} \times \mathbb{N}$ with the dictionary order. Then $(1, 1)$ is the least element of $X$, though there is no greatest element of $X$. The ordering produces the inequalities: $(1, 1) < (1, 2) < (1, 3) < \cdots < (2, 1) < (2, 2) < \cdots$ where the first “$\cdots$” indicates that all elements of the form $(1, n)$ are present. Notice that all but one singleton is in the basis $\mathcal{B}$ for the order topology:

- $(1, 1) = [(1, 1), (1, 2))$,
- $(1, n) = (1, n - 1), (1, n + 1))$ for $n > 1$,
- $(2, n) = (2, n - 1), (2, n + 1))$ for $n > 1$,

but a basis element containing $(2, 1)$ must be of the form $(a, b)$ where $a < (2, 1)$ and $(2, 1) < b$. But then $a$ is of the form $(1, n)$ for some $n \in \mathbb{N}$, so $(a, b)$ contains an infinite number of elements of $X$ less than $(2, 1)$. Now any open set containing $(2, 1)$ must contain a basis element about $(2, 1)$ and so the singleton $(2, 1)$ is the lone singleton in the topological space which is not open.

**Definition.** If $X$ is a set with a simple order relation $<$, and $a \in X$ then there are four subsets of $X$, called *rays* determined by $a$. They are the following:

- $(a, +\infty) = \{x \in X \mid x > a\}$
- $(-\infty, a) = \{x \in X \mid x < a\}$
- $[a, +\infty) = \{x \in X \mid x \geq a\}$
- $(-\infty, a) = \{x \in X \mid x \leq a\}$.

The first two types of rays are called *open rays* and the last two types are called *closed rays*. 
Note. The open rays in $X$ are open sets in the order topology since:

(1) If $X$ has a greatest element $b_0$ then $(a, +\infty) = (a, b_0] \in \mathcal{B}$ is given.

(2) If $X$ does not have a greatest element then $(a, +\infty) = \sup_{x > a} (a, x)$ is open.

(3) If $X$ has a least element $a_0$ then $(-\infty, a) = [a_0, a) \in \mathcal{B}$ is open.

(4) If $X$ does not have a least element then $(-\infty, a) = \bigcup_{x < a} (x, a)$ is open.

Notice that we have not yet defined “closed set,” but we will in Section 17.

**Theorem 14.B.** Let $X$ be a set with a simple order relation. The open rays form a subbasis for the order topology $\mathcal{T}$ on $X$. 

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