Section 28. Limit Point Compactness

Note. In this brief section we introduce two properties equivalent to compactness for metrizable spaces. One of the properties is stronger than compactness in a more general setting. We also introduce other examples of a nonmetrizable space.

Definition. A space $X$ is limit point compact if every infinite subset of $X$ has a limit point.

Note. The term “limit point compact” is due to Munkres (see page 179, line 3). Munkres comments that the property is sometimes also called “Fréchet compactness” or the “Bolzano-Weierstrass property.” Recall that the Bolzano-Weierstrass Theorem states that a bounded infinite set of real numbers (or elements of $\mathbb{R}^n$) must have a limit point.

Theorem 28.1. Compactness implies limit point compactness, but not conversely.

Example 1. Let $Y = \{y_1, y_2\}$ and let the topology on $Y$ consist of $\emptyset$ and $Y$. Consider $X = \mathbb{N} \times Y$ with the product topology where $\mathbb{N}$ has the discrete topology. Then for $A \subset X$, $A \neq \emptyset$, $A$ has an element of the form $(n, y_i)$. Any open set containing $(n, y_i)$ also contains the point $(n, y_j)$ where $j = 3 - i$, and so every nonempty $A \subset X$ has a limit point and so $X$ is limit point compact. However, $X$ is not compact since the covering of $X$ by the open sets $U_n = \{n\} \times Y$ for $n \in \mathbb{N}$ has no subcollection covering $X$ (and so no finite subcollection covering $X$).
**Definition.** Let $X$ be a topological space. If $\{x_n\}_{n=1}^\infty$ is a sequence of points in $X$ and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of natural numbers, then the sequence $\{y_i\}_{i=1}^\infty$ defined as $y_i = x_{n_i}$ is a subsequence of the sequence $\{x_n\}$. The space $X$ is *sequentially compact* if every sequence of points of $X$ has a convergent subsequence.

**Note.** We now show that each of three three types of compactness are the same in metrizable spaces. We follow Munkres’ proof, but break it into pieces, including two preliminary lemmas.

**Lemma 28.A.** Let $X$ be metrizable. If $X$ is also sequentially compact then the conclusion of the Lebesgue Number Lemma (Lemma 27.5) holds for $X$.

**Lemma 28.B.** Let $X$ be metrizable. If $X$ is also sequentially compact, then for all $\varepsilon > 0$ there exists a finite covering of $X$ by open $\varepsilon$-balls.

**Theorem 28.2.** Let $X$ be a metrizable space. Then the following are equivalent:

1. $X$ is compact.
2. $X$ is limit point compact.
3. $X$ is sequentially compact.
Note. Munkres gives another “less trivial” example of a space which is limit point compact but not compact. We review some definitions from set theory before stating the example.

Definition. (From Section 3) A relation $C$ on a set $A$ is an ordered relation (or simple order or linear order) if:

1. (Comparability) For all $x, y \in A$ for which $x \neq y$, either $xCy$ or $yCx$.
2. (Nonreflexivity) For no $x \in A$ does $xCx$ hold.
3. (Transitivity) If $xCy$ and $yCx$ then $xCz$.

Example. $A = \mathbb{R}$ has order relations greater than $>$ and less than $<$. We could also take $A$ as $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$ under either $>$ or $<$.

Definition. (From Section 10) A set $A$ with order relation $<$ is well-ordered if every nonempty subset $A$ has a smallest element.

Example. $A = \mathbb{R}$ under the usual less than, $<$, is NOT well-ordered. $A = \mathbb{N}$ under the usual less than IS well-ordered.
Note. The Well-Ordering Principle (see page 65 of Section 10) states that every set $A$ has an order relation for which $A$ is well-ordered. In fact, this property is equivalent to the Axiom of Choice (which Munkres takes as given and so observes that the Well-Ordering Principle can be proved; that’s why Munkres calls it the “Well-Ordering Theorem”).

Definition. (From Section 10) Let $X$ be a well-ordered set. Given $\alpha \in X$, let $X_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}$. This is called the section of $X$ by $\alpha$.

Lemma 10.2. There exists a well-ordered set $A$ having a largest element $\Omega$, such that the section $S_\Omega$ of $A$ by $\Omega$ is uncountable but every other section of $A$ is uncountable.

Definition. For well-ordered set $A$ with largest element $\Omega$ as described in Lemma 10.2, the section $S_\Omega$ is a minimal uncountable well-ordered set. The well-ordered set $S_\Omega \cup \{\Omega\}$ is denoted $\overline{S}_\Omega$. We put the order topology on both $S_\Omega$ and $\overline{S}_\Omega$.

Theorem 10.3. If $A$ is a countable subset of $S_\Omega$, then $A$ has an upper bound in $S_\Omega$. 
Example 2a. We claim that the space \( S_\Omega \) is limit point compact. Let \( A \) be an infinite subset of \( S_\Omega \). Choose a countable infinite subset \( B \) of \( A \) (which can be done since the “smallest infinity” is countable infinite). Since \( B \subseteq S_\Omega \) is countable, by Theorem 10.3 set \( B \) has an upper bound \( b \) in \( S_\Omega \). Let \( a_0 \) be the smallest element of \( S_\Omega \) (which exists because \( S_\Omega \) is well-ordered). Then \( B \subseteq [a_0, b] \). \( S_\Omega \) has the least upper bound property by Exercise 10.1 (in fact, the exercise shows that every well-ordered set has the least upper bound property), so by Theorem 27.1 the interval \([a_0, b]\) is compact. By Theorem 28.1, \([a_0, b]\) is limit point compact and so set \( B \subseteq [a_0, b] \) has a limit point \( x \in [a_0, b] \). Then \( x \) is also a limit point of \( A \) (since \([a_0, b] \subseteq A\)). Therefore, \( S_\Omega \) is limit point compact.

Example 2b. We claim that \( S_\Omega \) is not compact. Munkres justifies this with the single claim that \( S_\Omega \) has no largest element.

Note. Since the two examples given above, \( X = \mathbb{N} \times \{y_1, y_2\} \) and \( X = S_\Omega \), are of spaces which are limit point compact but not compact then, by Theorem 28.2, we see that these spaces are not metrizable.

Example 3. We claim that \( \overline{S}_\Omega = S_\Omega \cup \{\Omega\} \) is not metrizable. Since \( S_\Omega = \{x \mid x \in A, x < \Omega\} \), then \( \Omega \in \overline{S}_\Omega \) is a limit point of \( S_\Omega \). But any sequence of elements of \( S_\Omega \) is bounded by an element of \( S_\Omega \), and so \( \Omega \) is not the limit of any sequence of elements of \( S_\Omega \). By The Sequence Lemma (lemma 21.2), \( \overline{S}_\Omega \) is not metrizable (the “converse” part).

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