Section 44. A Space-Filling Curve

Note. In this section, we give the surprising result that there is a continuous function from the interval $[0, 1]$ onto the unit square $[0, 1] \times [0, 1]$.

Note. We first refer to Hans Sagan’s *Space-Filling Curves* (Springer-Verlag, Universitext series, 1994) for some historical comments.

Note. In 1878, George Cantor proved that the interval $[0, 1]$ and the square $[0, 1] \times [0, 1]$ have the same cardinality so that there is a one to one and onto function from $[0, 1]$ to $[0, 1] \times [0, 1]$. In 1879, E. Netto proved that such a function must be discontinuous (in “Beitrag zur Mannigfaltigkeitslehre,” *Crelle Journal*, 86, 265–268 (1879)). In 1890, Guiseppe Peano (1858–1912) constructed a continuous onto mapping from $[0, 1]$ to $[0, 1] \times [0, 1]$ (“Sur une courbe, qui remplit toute une aire plane,” *Mathematische Annalen*, 36(1), 157–160 (1890)). The result image of $[0, 1]$ is called a “space-filling curve” (or “Peano curve”) and satisfies the surprising property that the one-dimensional interval is continuously mapped onto the two-dimensional square! Additional examples were given by David Hilbert (1862–1943), Eliakim H. Moore (1862–1932), Henri Lebesgue (1875–1941), Waclaw Sierpiński (1882–1969), George Pólya (1887–1985) and others [Sagan, page 1]. The references for these works are:


**Note.** The exercises in this section have you show:

1. There is a continuous functions from \([0,1]\) onto \([0,1]^n\) for any given \(n \in \mathbb{N}\) (Exercise 44.1).

2. There is a continuous function from \(\mathbb{R}\) onto \(\mathbb{R}^n\) for any give \(n \in \mathbb{N}\) (Exercise 44.2).

3. There is not a continuous function from \(\mathbb{R}\) onto \(\mathbb{R}^\omega\) where \(\mathbb{R}^\omega\) is given the product topology (Exercise 44.3).

**Note.** The Hahn-Mazurkiewicz Theorem states that: “A non-empty Hausdorff topological space is a continuous image of the unit interval if and only if it is a compact, connected, weakly locally connected, metrizable space.” It is named for
Stefan Mazurkiewicz (1888–1945) and Hans Hahn (1879–1934). A Hausdorff space that is the continuous image of the closed unit interval is called a Peano space. See Exercise 44.4 for a little more information.

**Theorem 44.1.** Let $I = [0, 1]$. There exists a continuous map $f : I \to I^2$ whose image fills up the entire square $I^2$ (that is, $f$ is onto).

**Proof.** We follow Munkres’ 4-step proof. In Steps 1 and 2 we define a sequence of continuous functions $(f_n)$ where $f_n : I \to I^2$. In Step 3 we show that $(f_n)$ is a Cauchy sequence and use the results of Section 43 to show that $(f_n)$ converges to a continuous function $f : I \to I^2$. In Step 4 we show that $f$ is onto.

**Step 1.** First, we map $I$ into $I^2$ with $f_0 = g$ as shown in Figure 44.1. It is easy to express $f_0$ parametrically as

$$f_0(t) = \begin{cases} (t, t) & \text{for } t \in [0, 1/2] \\ (t, 1-t) & \text{for } t \in (1/2, 1]. \end{cases}$$

We then modify $f_0 = g$ to produce $f_1 = g'$ as shown in Figure 44.2. Partition $I$ into 8 pieces and partition $I^2$ into 4 pieces such that $f_1 = g'$ behaves similarly on the upper two squares and $f_1 = g'$ behaves on the lower two squares as shown. We could describe $f_1 = g'$ piecewise using 8 pieces.
Step 2. We now partition each square of Figure 44.1 into 4 subsquares and produce $f_2$ as given in Figure 44.4. Each square is Figure 44.4 is further partitioned into 4 squares and $f_3$ produced as shown in Figure 44.5 (there are 64 subsquares in Figure 44.5, though only 16 of them are shown). The iterative process leads us to 4 cases in terms of how to define $f_{n+1}$, in terms of $f$. One of the cases is given in how $g'$ is produced from $g$ in Figures 44.1 and 44.2. The other three cases are symmetries of this case. If a square has $f_n$ defined on it as given in Figure 44.1 rotated $180^\circ$ (with the segment of $f_n$ on this square starting and ending at the upper corners of the squares) then $f_{n+1}$ on the subsquares is given in Figure 44.2 rotated $180^\circ$.

If a square has $f_n$ defined as in Figure 44.3 then $f_{n+1}$ is defined on the 4 subsquares as given by $h'$ in Figure 44.3. If a square has $f_n$ defined as it is given in Figure 44.3 rotated $180^\circ$ (with the segment of $f_n$ on the square starting and ending on the right corners of the square) then $f_{n+1}$ on the subsquares is given by Figure 44.3 rotated $180^\circ$. Notice that $f_n$ is defined on $4^n$ subsquares, each containing 2 linear pieces of $f_n$ (so $f_n$ consists of $2 \times 4^n$ linear pieces) and that the length of a side of each subsquare is $1/2^n$. 
Step 3. Let \( d(x, y) \) denote the square metric on \( \mathbb{R}^2 \):
\[
    d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}
\]
(a metric on \( \mathbb{R}^n \) introduced in Section 20). Let \( \rho \) denote the corresponding sup metric on \((I, I^2)\):
\[
    \rho(f, g) = \sup\{d(f(t), g(t)) \mid t \in I\}.
\]
By Theorem 43.2, \( \mathbb{R}^2 \) is complete under \( \rho \). Since \( I^2 \) is closed in \( \mathbb{R}^2 \), then any Cauchy sequence in \( I^2 \) converges in \( \mathbb{R}^2 \), but the limit of the Cauchy sequence is a limit point of \( I^2 \) and since \( I^2 \) is closed, then the limit is in \( I^2 \). That is, any Cauchy sequence in \( I^2 \) converges in \( I^2 \). Hence \( I^2 \) is complete under \( \rho \). By Theorem 43.6, \( C(I, I^2) \) is complete in the metric \( \bar{\rho} \). Since a sequence is Cauchy under \( \rho \) if and only if it is Cauchy under \( \bar{\rho} \) (see the note on page 264 or the “Note” in the class notes before Lemma 43.1), then \( C(I, I^2) \) is complete in the metric \( \rho \).

We claim that \((f_n)\) is defined piecewise on \(4^n\) squares each with a side of length \(1/2^n\). Since \( f_{n+1} \) is based on \( f_n \) and each square is partitioned into 4 subsquares, then \( f_n \) under the square metric differed on each subsquare by at most \(1/2^n\). So \( \rho(f_n, f_{n+1}) \leq 1/2^n \). So by the Triangle Inequality
\[
    \rho(f_n, f_{n+m}) \leq \rho(f_n, f_{n+1}) + \rho(f_{n+1}, f_{n+2}) + \cdots + \rho(f_{n+m-1}, f_{n+m})
\]
\[
    \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m-1}} < \frac{1}{2^n} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} + \cdots = \frac{1/2^n}{1 - 1/2} = \frac{2}{2^n} = \frac{1}{2^{n-1}}
\]
for all \( m, n \in \mathbb{N} \). So for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( 1/2^{n-1} < \varepsilon \) and so for all \( m, n \geq N \) we have \( \rho(f_n, f_{n+m}) < \varepsilon \). That is, \((f_n)\) is a Cauchy sequence with respect to \( \rho \). Since \( C(I, I^2) \) is complete, then \((f_n)\) converges in \( C(I, I^2) \) and so there is continuous \( f : I \to I^2 \) such that \((f_n) \to f \).
Step 4. Finally, we show that $f$ is onto. Let $x \in I^2$. For given $n \in \mathbb{N}$, $x$ is in some subsquare with side of length $1/2^n$ and so there is a point $t_0 \in I$ such that $d(f(t_0), x) \leq 1/2^n$. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $1/2^N < \varepsilon/2$ and so there is $t_0 \in I$ such that $d(f(t_0), x) < \varepsilon$. Therefore $x$ is a limit point of $f(I)$. Since $f$ is continuous and $I$ is compact (by Corollary 27.2) then $f(I)$ is compact by Theorem 26.5. By Theorem 26.3, $f(I)$ is therefore closed and so by Theorem 17.6 $f(I)$ includes its limit points. That is, $x \in F(I)$ and so $f$ is onto, as claimed.

\[\square\]