Section I.8. Direct Products and Direct Sums

Note. In this section, we introduce general products of groups. This idea will be important when we classify finitely generated abelian groups in Section II.2.

Note. We defined the direct product of two groups in Section I.1. We now extend this idea to a collection of groups indexed by an arbitrary index set $I$ (possibly infinite or maybe even uncountable).

Definition. Consider an indexed family of groups $\{G_i \mid i \in I\}$. Define a binary operation on the Cartesian product of sets $\prod_{i \in I} G_i$ as follows. If $f, g \in \prod_{i \in I} G_i$ (that is, $f, g : I \to \bigcup_{i \in I} G_i$ and $f(i), g(i) \in G_i$ for all $i \in I$) then $fg : I \to \prod_{i \in I} G_i$ is the function given by $i \mapsto f(i)g(i) \in G_i$. So $fg \in \prod_{i \in I} G_i$. The set $\prod_{i \in I} G_i$ together with this binary operation (where we identify $f \in \prod_{i \in I} G_i$ with its image, the ordered set $\{a_i \mid i \in I\}$) is the direct product (or complete direct sum) of the family of groups $\{G_i \mid i \in I\}$.

Note. In a crude sense, we can think of the elements of $\prod G_i$ as $|I|$-tuples in which the binary operation is performed componentwise. If $I = \{1, 2, \ldots, n\}$ is finite, then we denote $\prod G_i$ as $G_1 \times G_2 \times \cdots \times G_n$ (in multiplicative notation) or $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ (in additive notation).
Theorem I.8.1. If \( \{G_i \mid i \in I\} \) is a family of groups, then

(i) the direct product \( \prod G_i \) is a group,

(ii) for each \( k \in I \), the map \( \pi_k : \prod G_i \to G_k \) given by \( f \mapsto f(k) \) is an epimorphism (i.e., an onto homomorphism) of groups.

The map \( \pi_k \) is called the *canonical projection* of the direct product.

**Proof.** A homework exercise.

Theorem I.8.2. Let \( \{G_i \mid i \in I\} \) be a family of groups, let \( H \) be a group, and let \( \{\varphi_i : H \to G_i \mid i \in I\} \) a family of group homomorphisms. Then there is a unique homomorphism \( \varphi : H \to \prod G_i \) such that \( \pi_i \varphi = \varphi_i \) for all \( i \in I \) and this property determines \( \prod G_i \) uniquely up to isomorphism. (In other words, \( \prod G_i \) is a product in the category of groups.)

**Note.** If each \( G_i \) is abelian, then \( \prod G_i \) is abelian (since the operation on \( \prod G_i \) is calculated “component-wise” as given in the first definition of the notes for this section). So \( \prod G_i \) is a product in the category of abelian groups.

Definition I.8.3. The (*external*) weak direct product of a family of groups \( \{G_i \mid i \in I\} \), denoted \( \prod_{i \in I}^{w} G_i \), is the set of all \( f \in \prod_{i \in I} G_i \) such that \( f(i) = e_i \) (where \( e_i \in G_i \) is the identity element of \( G_i \)) for all but a finite number of \( i \in I \). If all the groups \( G_i \) are additive abelian groups, then \( \prod_{i \in I}^{w} G_i \) is called the (*external*) direct sum and is denoted \( \sum_{i \in I} G_i \).
Note. Of course if $I$ is finite, then there is no difference in the weak direct product and the direct product.

**Theorem I.8.4.** If $\{G_i \mid i \in I\}$ is a family of groups, then

(i) $\prod_{i \in I}^w G_i$ is a normal subgroup of $\prod_{i \in I} G_i$;

(ii) for each $k \in I$, the map $\iota_k : G_k \to \prod_{i \in I}^w G_i$ given by $\iota_k(a) = \{a_i\}_{i \in I}$ where $a_i = e_i$ for $i \neq k$ and $a_k = a$, is a monomorphism (one to one homomorphism) of groups;

(iii) for each $i \in I$, $\iota_i(G_i)$ is a normal subgroup of $\prod_{i \in I} G_i$.

**Proof.** A homework exercise.

**Definition.** The maps $\iota_k$ of Theorem I.8.4 are called the **canonical injections** of $G_k$ into $\prod_{i \in I}^w G_i$ (or $\prod_{i \in I} G_i$).

**Theorem I.8.5.** Let $\{A_i \mid i \in I\}$ be a family of (additive) abelian groups. If $B$ is an abelian group and $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism mapping the (external) direct sum $\sum A_i$ to $B$, $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i$ for all $i \in I$ and this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. That is, $\sum_{i \in I} A_i$ is a coproduct in the category of abelian groups.
Note. Notice the use of the homomorphism properties of $\psi$ and $\xi$ in the proof of Theorem I.8.5 would not work unless the homomorphisms are applied to finite sums.

Note. Theorem I.8.5 is false if we remove the restriction that the groups are abelian. So the (external) weak direct product (or sum) is not in general a coproduct in the category of all groups (as you will show in Exercise I.8.4).

Note. The following result gives conditions under which a group is isomorphic to the weak direct product of a family of its subgroups (notice that we again see the use of normal subgroups).

**Theorem I.8.6.** Let \( \{N_i \mid i \in I\} \) be a family of normal subgroups of a group \( G \) such that

(i) \( G = \langle \bigcup_{i \in I} N_i \rangle \);

(ii) for each \( k \in I \), we have \( N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \langle e \rangle \).

Then \( G \cong \prod_{i \in I}^w N_i \).

**Corollary I.8.7.** If \( N_1, N_2, \ldots, N_r \) are normal subgroups of a group \( G \) such that \( G = N_1N_2\cdots N_r \) and for each \( 1 \leq k \leq r \) we have \( N_k \cap (N_1 \cdots N_{k-1}N_{k+1} \cdots N_r) = \langle e \rangle \) then \( G \cong N_1 \times N_2 \times \cdots \times N_r \).
Proof. This follows from Theorem I.8.6 when we observe that \( \langle N_1 \cup N_2 \cup \cdots \cup N_r \rangle = N_1 N_2 \cdots N_r = \{n_1 n_2 \cdots n_r \mid n_i \in N + i\} \) by Theorem I.5.3 (plus mathematical induction).

Definition I.8.8. Let \( \{N_i \mid i \in I\} \) be a family of normal subgroups of a group \( G \) such that \( G = \langle \bigcup_{i \in I} N_i \rangle \) and for each \( k \in I \) we have \( N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \langle e \rangle \). Then \( G \) is an internal weak direct product of the family \( \{N_i \mid i \in I\} \) (or the internal direct sum if \( G \) is additive and abelian).

Note. The following (a corollary of Theorem I.8.6) classifies internal direct products.

Theorem I.8.9. Let \( \{N_i \mid i \in I\} \) be a family of normal subgroups of a group \( G \). \( G \) is the internal weak direct product of the family \( \{N_i \mid i \in I\} \) if and only if every nonidentity element of \( G \) is a unique product \( a_{i_1} a_{i_2} \cdots a_{i_n} \) with \( i_1, i_2, \ldots, i_n \) distinct elements of \( I \) and \( e \neq a_{i_k} \in N_{i_k} \) for each \( k = 1, 2, \ldots, n \).

Proof. A homework exercise.

Note. There is a subtle difference between internal and external weak direct products. For \( G \) an internal weak direct product of groups \( N_i \) we have by definition that each \( N_i \) is a subgroup of \( G \) and \( G \) is isomorphic to the external direct product \( \prod^w N_i \) by Theorem I.8.6. But the external direct product does not contain groups
$N_i$ but contains isomorphic copies of them. Elements of $\prod^w N_i$ are "$|I|$-tuples." The $|I|$-tuple with each entry $e$ except the $i$th entries range over the elements of $N_i$ forms a subgroup of $\prod^w N_i$ which is isomorphic to $N_i$. This subgroup is $\iota_i(N_i)$. For this reason, we write $G = \prod^w N_i$ when $G$ is isomorphic to the weak direct product of $\{N_i \mid i \in I\}$.

**Theorem I.8.10.** Let $\{f_i : G_i \to H_i \mid i \in I\}$ be a family of homomorphisms of groups and let $f = \prod f_i$ be the mapping of $\prod G_i \to \prod H_i$ given by $\{a_i\} \to \{f_i(a_i)\}$. Then $f$ is a homomorphism of groups such that $f(\prod G_i) = \prod^w H_i$, $\text{Ker}(f) = \prod \text{Ker}(f_i)$, and $\text{Im}(f) = \prod \text{Im}(f_i)$ (where all products are over $i \in I$). Consequently $f$ is a monomorphism (or epimorphism) if and only if each $f_i$ is a monomorphism (or epimorphism).

**Proof.** A homework exercise.

**Corollary I.8.11.** Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that $N_i$ is a normal subgroup of $G_i$ for each $i \in I$.

(i) $\prod N_i$ is a normal subgroup of $\prod G_i$ and $\prod G_i / \prod N_i \cong \prod G_i / N_i$.

(ii) $\prod^w N_i$ is a normal subgroup of $\prod^w G_i$ and $\prod^w G_i / \prod^w N_i \cong \prod^w G_i / N_i$.

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