Supplement. A Proof of The Snake Lemma

Supplement. The Snake Lemma. Let $R$ be a ring and

\[
\begin{array}{ccl}
A & \rightarrow & B \\
\alpha \downarrow & & \beta \downarrow \\
0 & \rightarrow & A' \rightarrow B' \rightarrow C'
\end{array}
\]

a commutative diagram of $R$-modules and $R$-module homomorphisms such that each row is an exact sequence. Then there is an exact sequence

\[
\text{Ker}(\alpha) \rightarrow \text{Ker}(\beta) \rightarrow \text{Ker}(\gamma) \xrightarrow{\delta} \text{Coker}(\alpha) \rightarrow \text{Coker}(\beta) \rightarrow \text{Coker}(\gamma).
\]

If in addition, $f_A : A \rightarrow B$ is a monomorphism then so is the homomorphism

\[
k_\alpha : A' \rightarrow B',
\]

and if $g_{B'} : B' \rightarrow C'$ is an epimorphism then so is $b_\beta : \text{Coker}(\beta) \rightarrow \text{Coker}(\gamma)$. Under these added conditions, we can extend the exact sequence on the left to include “$0 \rightarrow$” and on the right to include “$\rightarrow 0$.”

Proof. This proof is based on the online document [http://aix1.uottawa.ca/~rblute/COURSE2/SnakeLemma.pdf](http://aix1.uottawa.ca/~rblute/COURSE2/SnakeLemma.pdf), posted by Richard Blute of the University of Ottawa, but we largely use different symbols here. We present the proof in several steps; at each new step, we reuse symbols in new (but similar) roles.

First, let $f_A : A \rightarrow B$, $f_B : B \rightarrow C$, $g_A : A' \rightarrow B'$, and $g_{B'} : B' \rightarrow C'$ be the $R$-module homomorphisms. Define $k_\alpha : \text{Ker}(\alpha) \rightarrow \text{Ker}(\beta)$ and $k_\beta : \text{Ker}(\beta) \rightarrow \text{Ker}(\gamma)$ as the restricted functions $f_A|_{\text{Ker}(\alpha)}$ and $f_B|_{\text{Ker}(\beta)}$, respectively. Notice that if $a \in \text{Ker}(\alpha)$ then

\[
\beta(f_A(a)) = g_A'(\alpha(a)) \text{ since the diagram is commutative}
\]

\[
= g_A'(a)(0) = 0 \text{ since } g_A' \text{ is a homomorphism},
\]

so in fact $k_\alpha = f_A|_{\text{Ker}(\alpha)}$ does map $\text{Ker}(\alpha)$ to $\text{Ker}(\beta)$ (and similarly $k_\beta = f_B|_{\text{Ker}(\beta)}$ maps as claimed). So the given diagram including the $R$-module homomorphisms is:
Now, \( \text{Coker}(\alpha) = A' / \text{Im}(\alpha) \) and \( \text{Coker}(\beta) = B' / \text{Im}(\beta) \) by definition of “cokernel.” Define \( c_\alpha : \text{Coker}(\alpha) \to \text{Coker}(\beta) \) and \( c_\beta : \text{Coker}(\beta) \to \text{Coker}(\gamma) \) as

\[
c_\alpha(a' + \text{Im}(\alpha)) = g_{A'}(a') + \text{Im}(\beta) \quad \text{and} \quad c_\beta(b' + \text{Im}(\beta)) = g_{B'}(b') + \text{Im}(\gamma).
\]

We now show \( c_\alpha \) and \( c_\beta \) are well-defined. If \( a'_1 + \text{Im}(\alpha) = a'_2 + \text{Im}(\alpha) \), then \( a'_1 - a'_2 \in \text{Im}(\alpha) \) so that \( a'_1 - a'_2 = \alpha(a) \) for some \( a \in A \). Then

\[
\begin{align*}
g_{A'}(a'_1 - a'_2) &= g_{A'}(\alpha(a)) = (g_{A'} \circ \alpha)(a) \\
&= (\beta \circ f_A)(a) \quad \text{since the diagram is commutative} \\
&= \beta(f_A(a)) \in \text{Im}(\beta),
\end{align*}
\]

so

\[
c_\alpha(a'_2 + \text{Im}(\alpha)) = g_{A'}(a'_2) + \text{Im}(\beta) = g_{A'}(a'_2) + g_{A'}(a'_1 - a'_2) + \text{Im}(\beta) \\
= g_{A'}(a'_2) + g_{A'}(a'_1) - g_{A'}(a'_2) + \text{Im}(\beta) = g_{A'}(a'_1) + \text{Im}(\beta) = c_\alpha(a'_1 + \text{Im}(\alpha))
\]

and so \( c_\alpha \) is well-defined (that is, independent of the representative of the coset used). Similarly, \( c_\beta \) is well-defined. The sequence to be shown exact is then:

\[
\text{Ker}(\alpha) \xrightarrow{k_\alpha} \text{Ker}(\beta) \xrightarrow{k_\beta} \text{Ker}(\gamma) \xrightarrow{\delta} \text{Coker}(\alpha) \xrightarrow{\kappa_\alpha} \text{Coker}(\beta) \xrightarrow{\kappa_\beta} \text{Coker}(\gamma).
\]

We now define \( \delta : \text{Ker}(\gamma) \to \text{Coker}(\alpha) \) (the “middle of the snake”). Let \( c \in \text{Ker}(\gamma) \). Since the first row is exact, then \( f_B \) is an epimorphism ("onto"); because \( f_C : C \to \{0\} \) implies \( \text{Im}(f_B) = \text{Ker}(f_C) = C \), and so there is some \( b \in B \) such that \( f_B(b) = c \). Now \( \beta(b) \in B' \). Since \( c \in \text{Ker}(\gamma) \) then

\[
\begin{align*}
g_{B'}(\beta(b)) &= (g_{B'} \circ \beta)(b) \\
&= (\gamma \circ f_B)(b) \quad \text{since the diagram is commutative} \\
&= \gamma(f_B(b)) = \gamma(c) \quad \text{since } f_B(b) = c \\
&= 0 \quad \text{since } c \in \text{Ker}(\gamma).
\end{align*}
\]

So \( \beta(b) \in \text{Ker}(g_{B'}) = \text{Im}(f_{A'}) \) since the second row is exact. So \( \beta(b) \in \text{Ker}(g_{B'}) = \text{Im}(f_{A'}) \) since the second row is exact. So \( \beta(b) = g_{A'}(a') \) for some \( a' \in A' \). Introducing \( g_0 : \{0\} \to A' \) (the inclusion
map) and using the fact that the second row is exact, $\text{Im}(g_b) = \text{Ker}(g_{A'}) = \{0\}$ and so $g_{A'}$ is a monomorphism (one to one) by Theorem I.2.3 (see also the comment on page 170) and so $a'$ is the unique element of $A'$ such that $\beta(b) = g_{A'}(a')$. Now define

$$\delta(c) = a' + \text{Im}(\alpha).$$

Now we have determined $a'$ as follows:

$$f_B(b) = c \text{ for some } b \in B$$
$$\beta(b) = g_{A'}(a') \text{ for unique } a' \in A' \text{ (unique for given $b$).} \quad (*$$

So we have $\delta(c) = (g_{A'})^{-1}(\beta(b)) + \text{Im}(\alpha)$ where $b$ is some element of the inverse image of $\{c\}$ under $f_B$, $b \in f_B^{-1}\{c\}$. So to show that $\delta$ is well-defined (this is the topic of discussion between Dr. Kate Gunzinger and Mr. Cooperman in the 1980 Rastar Films’ It’s My Turn) we need to consider the value of $\delta(c)$ for two different elements of $f_B^{-1}\{c\}$, say both $b_1$ and $b_2$ satisfy $f_B(b_1) = c$ and $f_B(b_2) = c$. As above, there are unique $a'_1$ and $a'_2$ such that $g_{A'}(a'_1) = \beta(b_1)$ and $g_{A'}(a'_2) = \beta(b_2)$. Notice that $f_B(b_1 - b_2) = f_B(b_1) - f_B(b_2) = c - c = 0$ so that $b_1 - b_2 \in \text{Ker}(f_B)$. Since the first row is exact then $\text{Im}(f_A) = \text{Ker}(f_B)$, there is $a \in A$ such that $f_A(a) = b_1 - b_2$. Hence,

$$g_{A'}(\alpha(a)) = (g_{A'} \circ \alpha)(a)$$
$$= (\beta \circ f_A)(a) \text{ since the diagram is commutative}$$
$$= \beta(f_A(a)) = \beta(b_1 - b_2) = \beta(b_1) - \beta(b_2)$$
$$= g_{A'}(a'_1) - g_{A'}(a'_2) = g_{A'}(a'_1 - a'_2).$$

Since $g_{A'}$ is a monomorphism, then $a'_1 - a'_2 = \alpha(a) \in \text{Im}(\alpha)$. Hence

$$a'_2 + \text{Im}(\alpha) = a'_2 + \alpha(a) + \text{Im}(\alpha) = a'_2 + (a'_1 - a'_2) + \text{Im}(\alpha) = a'_1 + \text{Im}(\alpha),$$

so $\delta$ is well-defined. We now show exactness and work left to right.

Since the top row is exact, then $\text{Im}(f_A) = \text{Ker}(f_B)$ and so $f_B \circ f_A$ is the zero function. Therefore $k_{\beta} \circ k_{\alpha} = f_B|_{\text{Ker}(\beta)} \circ f_A|_{\text{Ker}(\alpha)}$ is the zero function. So $\text{Im}(k_{\alpha}) \subset \text{Ker}(k_{\beta})$. Conversely, suppose $b \in \text{Ker}(\beta)$ with $k_{\beta}(b) = 0$ (that is, $b \in \text{Ker}(k_{\beta})$). Then $f_B(b) = 0$, so $b \in \text{Ker}(f_B) = \text{Im}(f_A)$ and hence $b = f_A(a)$ for some $a \in A$. We have

$$g_{A'}(\alpha(a)) = (g_{A'} \circ \alpha)(a)$$
$$= (\beta \circ f_A)(a) \text{ since the diagram is commutative}$$
$$= \beta(f_A(a)) = \beta(b) = 0 \text{ since $b \in \text{Ker}(\beta)$}.$$
Since $g_A$ is a monomorphism (one to one), then $\alpha(a) = 0$ and $a \in \text{Ker}(\alpha)$. So $k_\alpha(a) = f_A(a) = b \in \text{Im}(k_\alpha)$ and $\text{Ker}(k_\beta) \subset \text{Im}(k_\alpha)$. Hence $\text{Im}(k_\alpha) = \text{Ker}(k_\beta)$ and the sequence is exact at $\text{Ker}(\beta)$.

Let $b \in \text{Ker}(\beta)$ (Ker$(\beta)$ is the domain of $k_\beta$) so that $k_\beta(b)$ is an arbitrary element of $\text{Im}(k_\beta)$. Since $\beta(b) = 0$ then $\beta(b) = g_A(0)$ (since $g_A : A' \to B'$ is a homomorphism), so $\delta(k_\beta(b)) = 0 + \text{Im}(\alpha) = \text{Im}(\alpha) = 0 \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha)$ (since $\beta(b) = g_A(0)$ and $a' = 0$ in the notation of the definition of $\delta$). Since $b \in \text{Ker}(\beta)$ is arbitrary (and Ker$(\beta)$ is the domain of $k(\beta)$) then $\delta(k_\beta(b)) = (\delta \circ k_\beta)(b) = 0$ for all $b \in \text{Ker}(\beta)$, and so $\text{Im}(k_\beta) \subset \text{ker}(\delta)$. Conversely, suppose $c \in \text{Ker}(\gamma)$ (Ker$(\gamma)$ is the domain of $\delta$) and $c \in \text{Ker}(\delta)$. Then $c = f_B(b)$ for some $b \in B$ (since the first row is exact and $\text{Im}(f_B) = c$; that is, $f_B$ is an epimorphism because the kernel of the mapping $C \to \{0\}$ is all of $C$). By the definition of $\delta$, since $c \in \text{Ker}(\delta)$, we have

$$\delta(c) = \text{Im}(\alpha) = 0 \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha).$$

Since $c = f_B(b)$ then $\beta(b) = g_A(a')$ for some $a' \in A'$ (see $(*)$ above), and since $\delta(c) = a' + \text{Im}(\alpha) = \text{Im}(\alpha)$, it must be that $a' \in \text{Im}(\alpha)$. Say $a' = \alpha(a)$ for $a \in A$. Then

$$\beta(f_A(a)) = (\beta \circ f_A)(a)$$
$$= (g_A' \circ \alpha)(a) \text{ since the diagram is commutative}$$
$$= g_A'(\alpha(a)) = g_A'(a')$$
$$= \beta(b) \text{ since } \beta(b) = g_A'(a).$$

Since $\beta$ is a homomorphism, $0 = \beta(b) - \beta(f_A(a)) = \beta(b - f_A(a))$ and $b - f_A(a) \in \text{Ker}(\beta)$. Finally,

$$k_\beta(b - f_A(a)) = f_B(b - f_A(a)) \text{ since } b - f_A(a) \in \text{Ker}(\beta) \text{ and by the definition of } k_\beta \text{ as } f_B|_{\text{Ker}(\beta)}$$
$$= f_B(b) - (f_B \circ f_A)(a) \text{ since } f_B \text{ is a homomorphism}$$
$$= f_B(b) - 0 \text{ since } \text{Im}(f_A) = \text{Ker}(f_B) \text{ by the exactness of the first row}$$
$$= f_B(b) = c \text{ since } c = f_B(b).$$

That is, $c \in \text{Im}(k_\beta)$. Since $c$ is an arbitrary element of Ker$(\delta)$ (in the domain Ker$(\gamma)$ of $\delta$), then Ker$(\delta) \subset \text{Ker}(k_\beta)$. Hence $\text{Im}(k_\beta) = \text{Ker}(\delta)$ and the sequence is exact at Ker$(\gamma)$.

Let $c \in \text{Ker}(\gamma)$ so that $\delta(c)$ is an arbitrary element of $\text{Im}(\gamma)$. Since $f_B$ is an epimorphism by the exactness of the first row, then $c = f_B(b)$ for some $b \in B$. By $(*)$ above we have $\beta(b) = g_A'(a')$ for some $a' \in A'$. So by the definition of $\delta$, $\delta(c) = a' + \text{Im}(\alpha)$. Then

$$(c_\alpha \circ \delta)(c) = c_\alpha(\delta(c)) = c_\alpha(a' + \text{Im}(\alpha))$$
$$= g_A'(a') + \text{Im}(\beta) \text{ by the definition of } c_\alpha.$$
\[ \beta(b) + \text{Im}(\beta) \]
\[ = \beta(b) + \text{Im}(\beta) \quad \text{since } \beta(b)g_{A'}(a') \]
\[ = \text{Im}(\beta) \quad \text{since } \beta(b) \in \text{Im}(\beta) \]
\[ = 0 \in B'/\text{Im}(\beta). \]

Since \( c \) is an arbitrary element of \( \text{Ker}(\gamma) \) (the domain of \( \delta \)), then \( c \circ \delta \) is the zero function and \( \text{Im}(\delta) \subset \text{Ker}(c) \). Conversely, suppose \( a' + \text{Im}(\alpha) \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha) \) is in the kernel of \( c \).

Then
\[
c_\alpha(a' + \text{Im}(\alpha)) = g_{A'}(a') + \text{Im}(\beta) \quad \text{by the definition of } c_\alpha
\]
\[ = \text{Im}(\beta) = 0 \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha) \quad \text{since } a' + \text{Im}(\alpha) \in \text{Ker}(c), \]

and so \( g_{A'}(a') \in \text{Im}(\beta) \), say \( g_{A'}(a') = \beta(b) \) for some \( b \in B \). Let \( c = f_B(b) \). Then
\[
\gamma(c) = \gamma(f_B(b)) = (\gamma \circ f_B)(b)
\]
\[ = (g_{B'} \circ \beta)(b) \quad \text{since the diagram commutes}
\]
\[ = g_{B'}(\beta(b)) = g_{B'}(g_{A'}(a')) = 0 \quad \text{since } \text{Im}(g_{A'}) = \text{Ker}(g_{B'})
\]
\[ = 0 \in C'/\text{Im}(\gamma) = \text{Coker}(\gamma). \]

So \( a' + \text{Im}(\alpha) \in \text{Im}(\delta) \). Since \( a' + \text{Im}(\alpha) \) is an arbitrary element of \( \text{Ker}(c) \) then \( \text{Ker}(c) \subset \text{Im}(\delta) \).

Hence \( \text{Im}(\delta) = \text{Ker}(c) \) and the sequence is exact at \( \text{Coker}(\alpha) \).

Since the second row is exact, then \( \text{Im}(g_{A'}) = \text{Ker}(g_{B'}) \) and so \( g_{B'} \circ g_{A'} \) is the zero function. Therefore for \( a' + \text{Im}(\alpha) \in \text{Coker}(\alpha) \),
\[
(c_\beta \circ c_\alpha)(a' + \text{Im}(\alpha)) = c_\beta(c_\alpha(a' + \text{Im}(\alpha)))
\]
\[ = c_\beta(g_{A'}(a') + \text{Im}(\beta)) \quad \text{by the definition of } c_\alpha
\]
\[ = g_{B'}(g_{A'}(a')) + \text{Im}(\gamma) \quad \text{by the definition of } c_\beta
\]
\[ = (g_{B'} \circ g_{A'})(a') + \text{Im}(\gamma) = \text{Im}(\gamma) \quad \text{since } g_{B'} \circ g_{A'} \text{ is the zero function}
\]
\[ = 0 \in C'/\text{Im}(\gamma) = \text{Coker}(\gamma). \]

Since \( a' + \text{Im}(\alpha) \) is an arbitrary element of \( \text{Coker}(\alpha) \) (the domain of \( c_\beta \circ c_\alpha \)), then \( c_\beta \circ c_\alpha \) is the zero function and \( \text{Im}(c_\alpha) \subset \text{Ker}(c_\beta) \). Conversely, suppose \( b' + \text{Im}(\beta) \in B'/\text{Im}(\beta) = \text{Cojer}(\beta) \) is in \( \text{Ker}(c_\beta) \). Then
\[
c_\beta(b' + \text{Im}(\beta)) = g_{B'}(b') + \text{Im}(\gamma) = \text{Im}(\gamma) = 0 \in B'/\text{Im}(\beta) = \text{Coker}(\beta), \]
We started with \( g_{B'}(b') \in \text{Im}(\gamma) \), and so \( \gamma(c) = g_{B'}(b') \) for some \( c \in C \). Since \( f_B \) is an epimorphism by the exactness of the first row, then \( c = f_B(b) \) for some \( b \in B \). Now \( \beta(b) \in \text{Im}(\beta) \), so \( b' + \text{Im}(\beta) = b' - \beta(b) + \text{Im}(\beta) \) in \( B'/\text{Im}(\beta) = \text{Coker}(\beta) \). Now

\[
g_{B'}(b' - \beta(b)) = g_{B'}(b') - g_{B'}(\beta(b)) = g_{B'}(b') - (g_{B'} \circ \beta)(b)
\]

\[
= g_{B'}(b') - (\gamma \circ f_B)(b) \text{ since the diagram is commutative }
\]

\[
= \gamma(c) - \gamma(f_B(b)) \text{ since } \gamma(c) = g_{B'}(b')
\]

\[
= \gamma(c) - \gamma(c) = 0 \text{ since } f_B(b) = c.
\]

We started with \( b' + \text{Im}(\beta) = b' - \beta(b) + \text{Im}(\beta) \) as an arbitrary element of \( \text{Coker}(\beta) \) and saw that \( g_{B'}(b' - \beta(b)) = 0 \), so without loss of generality we can assume \( g_{B'}(b') = 0 \) (just replace \( b' \) with \( b' - \beta(b) \) as the representation of coset \( b' + \text{Im}(\beta) \)). That is, \( b' \in \text{Ker}(g_{B'}) \) without loss of generality. Since the second row is exact, then \( \text{Im}(g_A) = \text{Ker}(g_{B'}) \) and so \( b' \in \text{Im}(g_A) \). Hence \( b' = g_A(a') \) for some \( a' \in A' \). Then

\[
c_\alpha(a' + \text{Im}(\alpha)) = g_{A'} + \text{Im}(\beta) \text{ by the definition of } c_\alpha
\]

\[
= b' + \text{Im}(\beta),
\]

and so \( b' + \text{Im}(\beta) \in \text{Im}(c_\alpha) \). Since \( b' + \text{Im}(\beta) \) is an arbitrary element of \( \text{Ker}(c_\beta) \), then \( \text{Ker}(c_\beta) \subset \text{Im}(c_\alpha) \). Hence \( \text{Im}(c_\alpha) = \text{Ker}(c_\beta) \) and the sequence is exact at \( \text{Coker}(\beta) \). Therefore, the sequence

\[
\text{Ker}(\alpha) \xrightarrow{k_\alpha} \text{Ker}(\beta) \xrightarrow{k_\beta} \text{Ker}(\gamma) \xrightarrow{\delta} \text{Coker}(\alpha) \xrightarrow{c_\alpha} \text{Coker}(\beta) \xrightarrow{c_\beta} \text{Coker}(\gamma)
\]

is exact.

Finally, if \( f_A \) is a monomorphism (“one to one”; in which case the first row of the diagram can be extended to the left to include “\( 0 \rightarrow \)”), then \( k_\alpha = f_A|_{\text{Ker}(\alpha)} \) is a monomorphism, as claimed. The exact sequence of kernels and cokernels can then be extended to the left to include “\( 0 \rightarrow \)” If \( g_{B'} \) is an epimorphism (“onto”; in which case the second row of the diagram can be extended to the right to include “\( \rightarrow 0 \)” and \( c' + \text{Im}(\gamma) \in C'/\text{Im}(\gamma) = \text{Coker}(\gamma) \), then \( g_{B'}(b') = c' \) for some \( b' \in B' \). So

\[
c_\beta(b' + \text{Im}(\beta)) = g_B(b') + \text{Im}(\gamma) \text{ by the definition of } c_\beta
\]

\[
= c' + \text{Im}(\gamma),
\]

and \( c' + \text{Im}(\gamma) \in c_\beta \). Since \( c' + \text{Im}(\gamma) \) is an arbitrary element of \( \text{Coker}(\gamma) \), then \( \text{Im}(c_\beta) = \text{Coker}(\gamma) \) and \( c_\beta \) is an epimorphism (onto), as claimed. The exact sequence of kernels and cokernels then can be extended to the right to include “\( \rightarrow 0 \)”. \( \blacksquare \)