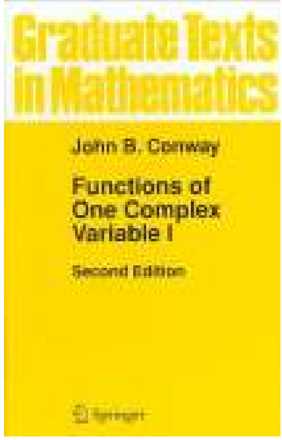
Complex Analysis

The Extended Complex Plane

Proofs of Theorems


Graduate Texts in Mathematics

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Functions of One Complex Variable I

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Topologies on $\mathbb{C}_\infty$ Theorem

Topologies on $\mathbb{C}_\infty$ Theorem.
Let $G \subset \mathbb{C}$. Then G is open in metric space $(\mathbb{C}, |\cdot|)$ if and only if G is open in metric space $(\mathbb{C}_\infty, d)$.

Proof. Let $G \subset \mathbb{C}$ be open in $(\mathbb{C}, |\cdot|)$ and let $a \in G$. Then, by the definition of "open," there is $r > 0$ such that $B(a; r) \subset G$. By Proposition VII.3.3(a), there is $\rho > 0$ such that $B_\infty(a, \rho) \subset B(a; r) \subset G$. Hence G is open in $(\mathbb{C}_\infty, d)$.

Let $G \subset \mathbb{C}$ be open in $(\mathbb{C}_\infty, d)$ and let $a \in G$. Then, by definition of "open," there is $\rho > 0$ such that $B_\infty(a, \rho) \subset G$. By Proposition VII.3.3(b), there is $r > 0$ such that $B(a; r) \subset B_\infty(a; \rho) \subset G$. Hence G is open in $(\mathbb{C}_\infty, |\cdot|)$. □

Sequences in $\mathbb{C}_\infty$ Theorem

Sequences in $\mathbb{C}_\infty$ Theorem.
Let $\{z_n\}_{n=1}^\infty$ be a sequence of complex numbers. Then for $z \in \mathbb{C}$, $z_n \to z$ in metric space $(\mathbb{C}, |\cdot|)$ if and only if $z_n \to z$ in metric space $(\mathbb{C}_\infty, d)$.

Proof. Suppose $z_n \to z$, where $z \in \mathbb{C}$, in $(\mathbb{C}, |\cdot|)$. Let $\varepsilon_2 > 0$. Then Proposition VII.3.3(b) there is $\varepsilon_1 > 0$ such that $B(z; \varepsilon_1) \subset B_\infty(z; \varepsilon_2)$. Since $z_n \to z$ in $(\mathbb{C}, |\cdot|)$, then there is $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|z_n - z| < \varepsilon_1$; that is, $n \geq N_1$ implies $z_n \in B(z; \varepsilon_1) \subset B_\infty(z; \varepsilon_2)$. Since $\varepsilon_2 > 0$ is arbitrary, we have that $z_n \to z$ in metric space $(\mathbb{C}_\infty, d)$.

Suppose $z_n \to z$, where $z \in \mathbb{C}$, in $(\mathbb{C}_\infty, d)$. Let $\varepsilon_1 > 0$. Then Proposition VII.3.3(b) there is $\varepsilon_2 > 0$ such that $B_\infty(z; \varepsilon_2) \subset B_\infty(z; \varepsilon_1)$. Since $z_n \to z$ in $(\mathbb{C}_\infty, d)$, then there is $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $d(z_n, z) < \varepsilon_2$; that is, $n \geq N_2$ implies $z_n \in B(z; \varepsilon_2) \subset B_\infty(z; \varepsilon_1)$. Since $\varepsilon_1 > 0$ is arbitrary, we have that $z_n \to z$ in metric space $(\mathbb{C}, |\cdot|)$. □

Compactness of $\mathbb{C}_\infty$ Theorem

Compactness of $\mathbb{C}_\infty$ Theorem.
$\mathbb{C}_\infty$ is a compact metric space under d.

Proof. Let $\mathcal{G}$ be a collection of open sets in metric space $(\mathbb{C}_\infty, d)$ such that $\mathbb{C}_\infty \subset \sup_{G \in \mathcal{G}} G$. Then $\infty \in G$ for some $G \in \mathcal{G}$, say $\infty \in G_0$. Since G_0 is open and $\infty \in G_0$, there is $\rho > 0$ such that $B_\infty(\infty, \rho) \subset G_0$. So by Proposition VII.2.2(c), there is a compact set K in metric space $(\mathbb{C}, |\cdot|)$ such that $\mathbb{C}_\infty \setminus K \subset B_\infty(\infty, \rho)$.

Next, for every $z \in K$ there is $G_z \in \mathcal{G}$ such that $z \in G_z$, since $\mathcal{G}$ is an open cover of $\mathbb{C}_\infty$. Since G_z is open in $(\mathbb{C}_\infty, d)$ then there is $\rho_z > 0$ such that $B_\infty(z, \rho_z) \subset G_z$. By Proposition VII.2.2(b), there is $r_z > 0$ such that $B(z, r_z) \subset B_\infty(z, \rho_z) \subset G_z$. Taking all such $B(z, r_z)$ for $z \in K$ gives an open cover of K: $K \subset \cup_{z \in K} B(z, r_z)$.

Compactness of $\mathbb{C}_\infty$ Theorem (continued)

Compactness of $\mathbb{C}_\infty$ Theorem.
$\mathbb{C}_\infty$ is a compact metric space under d.

Proof (continued). Since K is compact in $(\mathbb{C}, |\cdot|)$ then there are finitely many $z \in K$ such that the corresponding $B(z, t_z)$ cover K, say $K \subset B(z_1, r_{z_1}) \cup B(z_2, r_{z_2}) \cup \cdots \cup B(z_n, r_{z_n})$. Since $B(z, r_z) \subset G_z$ for all $z \in K$, then $K \subset G_{z_1} \cup G_{z_2} \cup \cdots \cup G_{z_n}$ and so $\{G_{z_i} \mid i = 1, 2, \ldots, n\}$ is an open cover of K. Therefore $\{G_0\} \cup \{G_i \mid i = 1, 2, \ldots, n\} \subset \mathcal{G}$ is a finite cover of $\mathbb{C}_\infty$ which is a subcover of $\mathcal{G}$ and so $\mathbb{C}_\infty$ is compact. $\square$