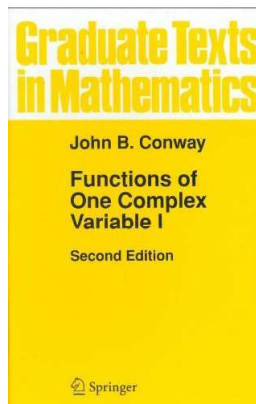


Complex Analysis

Chapter I. The Complex Number System I.3. The Complex Plane—Proofs of Theorems



The Triangle Inequality

Theorem. The Triangle Inequality.

For all $z, w \in \mathbb{C}$, $|z + w| \leq |z| + |w|$.

Proof. Let $z = a + ib$. Then

$$-|z| = -\sqrt{a^2 + b^2} \leq -\sqrt{a^2} = -|a| \leq \operatorname{Re}(z) \leq |a| \leq |z|.$$

So $-|z| \leq \operatorname{Re}(z) \leq |z|$ and $|\operatorname{Re}(z)| \leq |z|$. From Exercise I.2.4(a), $|z + w|^2 = |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2$, and so

$$\begin{aligned} |z + w|^2 &\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 \text{ since } |w| = |\bar{w}| \\ &= (|z| + |w|)^2. \end{aligned}$$

Therefore $|z + w| \leq |z| + |w|$. □

Corollary

Corollary

Corollary. For nonzero $z, w \in \mathbb{C}$, $|z + w| = |z| + |w|$ if and only if $z = tw$ for some $t \in \mathbb{R}$, $t \geq 0$.

Proof. First, if $z = tw$ then

$$\begin{aligned} |z + w| &= |tw + w| = |(t + 1)w| = |t + 1||w| \\ &= (t + 1)|w| = t|w| + |w| \\ &= |tw| + |w| = |z| + |w|. \end{aligned}$$

Second, we see in the proof of the Triangle Inequality that we get equality when $\operatorname{Re}(z\bar{w}) = |z\bar{w}|$. This only occurs when $\operatorname{Im}(z\bar{w}) = 0$ and $\operatorname{Re}(z\bar{w}) \geq 0$. That is, $z\bar{w} \geq 0$ or (if $w \neq 0$) we have $z\bar{w}(w/w) \geq 0$ or

$$\frac{z\bar{w}}{w} \geq 0 \text{ implies } \frac{z|w|^2}{w} = \frac{z}{w}|w|^2 = s \geq 0$$

where $s \in \mathbb{R}$. So we have that $z = (s/|w|^2)w = tw$ where $t = s/|w|^2 \geq 0$. □

Cauchy Sequences Theorem

Cauchy Sequences Theorem

Theorem. A Cauchy sequence of complex numbers is convergent.

Proof. Let (z_n) be a Cauchy sequence of complex numbers and let $\operatorname{Re}(z_n) = a_n$, $\operatorname{Im}(z_n) = b_n$ for each $n \in \mathbb{N}$. By definition of Cauchy sequence, we know that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have $|z_m - z_n| < \varepsilon$. Notice that

$$\begin{aligned} |z_m - z_n| &= |(a_m + ib_m) - (a_n + ib_n)| \\ &= |(a_m - a_n) + i(b_m - b_n)| = \sqrt{(a_m - a_n)^2 + (b_m - b_n)^2}. \end{aligned}$$

Now $|a_m - a_n| = \sqrt{(a_m - a_n)^2} \leq \sqrt{(a_m - a_n)^2 + (b_m - b_n)^2} = |z_m - z_n|$, so if $m, n \geq N$ then $|a_m - a_n| \leq |z_m - z_n| < \varepsilon$, and (a_n) is a Cauchy sequence of real numbers. Since the real numbers are complete, then Cauchy sequences converge and $(a_n) \rightarrow a$ for some $a \in \mathbb{R}$. Similarly, (b_n) is a Cauchy sequence of real numbers and $(b_n) \rightarrow b$ for some $b \in \mathbb{R}$.

Cauchy Sequences Theorem (continued)

Theorem. A Cauchy sequence of complex numbers is convergent.

proof (continued). We claim that $(z_n) \rightarrow a + ib$. Let $\varepsilon > 0$ be given. Since $(a_n) \rightarrow a$ and $(b_n) \rightarrow b$, there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|a_n - a| < \varepsilon/2$ and for all $n \geq N_2$ we have $|b_n - b| < \varepsilon/2$. So for all $n \geq \max\{N_1, N_2\}$ we have

$$\begin{aligned} |z_n - (a + ib)| &= |(a_n + ib_n) - (a + ib)| = |(a_n - a) + i(b_n - b)| \\ &\leq |a_n - a| + |i(b_n - b)| \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore $(z_n) \rightarrow a + ib$ and the Cauchy sequence of complex numbers is convergent. \square