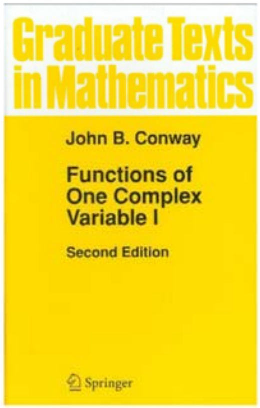
Complex Analysis

Chapter I. The Complex Number System

Supplement. Location of Zeros of Polynomials—Proofs of Theorems



Cauchy's Location of Zeros Theorem, Category (1)

Theorem 1. Cauchy's Location of Zeros Theorem, Category (1). If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \le 1 + \max_{0 \le k < n} |a_k/a_n| = \max_{0 \le k < n} \frac{|a_n| + |a_k|}{|a_n|}.$$

Proof. Let $M = \max_{0 \le k < n} |a_k/a_n|$. By the Triangle Inequality (and Exercise I.3.1),

$$\begin{aligned} |p(z)| &= \left| a_n z^n + (a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}) \right| \\ &\ge |a_n z^n| - |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}| \\ &\ge |a_n z^n| - (|a_0| + |a_1||z| + |a_2||z|^2 + \cdots + |a_{n-1}||z|^{n-1}). \qquad (*) \end{aligned}$$

Cauchy's Location of Zeros Theorem, Category (1) (continued 1)

Proof (continued). Therefore, for $|z| > 1$ we have

$$\begin{aligned} |p(z)| &\ge |a_n||z|^n - \left(\sum_{k=0}^{n-1} |a_k||z|^k \right) \text{ by } (*) \\ &= |a_n||z|^n \left(1 - \sum_{k=0}^{n-1} |a_k/a_n||z|^{k-n} \right) \\ &\ge |a_n||z|^n \left(1 - M \sum_{k=0}^{n-1} |z|^{k-n} \right) = |a_n||z|^n \left(1 - M \sum_{k=1}^{n} |z|^{-k} \right) \\ &\ge |a_n||z|^n \left(1 - M \sum_{k=1}^{\infty} |z|^{-k} \right) \end{aligned}$$

...

Cauchy's Location of Zeros Theorem, Category (1) (continued 2)

Proof (continued). ...

$$|p(z)| \ge |a_n||z|^n \left(1 - M \frac{1/|z|}{1 - 1/|z|} \right) \text{ since } \sum_{k=1}^{\infty} |z|^{-k} \text{ is a}$$

geometric series with ratio $1/|z| < 1$ and first term $1/|z|$

$$= |a_n||z|^n \left(1 - \frac{M}{|z| - 1} \right) = |a_n||z|^n \left(\frac{|z| - 1 - M}{|z| - 1} \right).$$

Hence, if $|z| > 1 + M$, then $|p(z)| > 0$ and so $p(z) \ne 0$. So all zeros of p in $|z| > 1$ must satisfy $|z| \le 1 + M$; of course the zeros of p in $|z| \le 1$ already lie in $|z| \le 1 + M$. Therefore, all zeros of p lie in $|z| \le 1 + M = 1 + \max_{0 \le k < n} |a_k/a_n|$ (notice that if $M = \max_{0 \le k < n} |a_k/a_n| = 0$ then $p(z) = a_n z^n$ and all zeros are at $z = 0$). □

Theorem 2. Cauchy's Location of Zeros Theorem, Category (2)

Theorem 2. Cauchy's Location of Zeros Theorem, Category (2). If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in $|z| \le r$, where r is the positive root of the equation

$$|a_n|x^n - (|a_{n-1}|x^{n-1} + |a_{n-2}|x^{n-2} + \cdots + |a_1|x + |a_0|) = 0.$$

Proof. As in the proof of Theorem 1, from the Triangle Inequality (and Exercise I.3.1) we have

$$|p(z)| \ge |a_n||z|^n - (|a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots |a_1||z| + |a_0|).$$

By Descartes' Rule of Signs, the equation

$$f(x) = |a_n|x^n - (|a_{n-1}|x^{n-1} + |a_{n-2}|x^{n-2} + \cdots |a_1|x + |a_0|) = 0$$

has exactly one real positive root r. Notice that for "large" positive x, $f(x) > 0$ and so $f(x) > 0$ for $x > r$. That is, $|p(z)| > 0$ (and hence $p(z) \ne 0$) for $|z| > r$. So all zeros of p lie in $|z| \le r$. □

Theorem 4

Theorem 4. Kuniyeda, Montel, and Tôya For any p and q such that $1/p + 1/q = 1$, $p > 1$, and $q > 1$, all zeros of polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ lie in

$$|z| < \left\{ 1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{q/p} \right\}^{1/q} \le \left(1 + n^{q/p} \left(\max_{0 \le k \le n-1} \frac{|a_k|}{|a_n|} \right)^q \right)^{1/q}.$$

Proof. We have

$$\begin{aligned} |p(z)| &\ge |a_n||z|^n - (|a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots + |a_1||z| + |a_0|) \\ &= |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \text{ by Triangle Inequality and Exercise I.3.1} \\ &\ge |a_n||z|^n - \left(\sum_{k=0}^{n-1} |a_k|^p \right)^{1/p} \left(\sum_{k=0}^{n-1} |z|^{kq} \right)^{1/q} \text{ by Hölder's Inequality} \end{aligned}$$

Theorem 4 (continued 1)

Proof (continued). ...

$$|p(z)| \ge |a_n||z|^n \left(1 - \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{1/p} \left(\sum_{k=0}^{n-1} \frac{1}{|z|^{(n-k)q}} \right)^{1/q} \right). \quad (*)$$

If $|z| > 1$ then

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{1}{|z|^{(n-k)q}} &= \sum_{k=1}^{n} \frac{1}{|z|^{kq}} \text{ replacing } k \text{ with } n-k \\ &< \sum_{k=1}^{\infty} \frac{1}{|z|^{kq}} \text{ under the assumption that } |z| > 1 \\ &= \sum_{k=0}^{\infty} \frac{1}{|z|^{kq}} - 1 = \frac{1}{1 - |z|^{-q}} - 1 \text{ since the series} \end{aligned}$$

is a geometric series with ratio $|z|^{-q} < 1$

Theorem 4 (continued 2)

Proof (continued). ...

$$\sum_{k=0}^{n-1} \frac{1}{|z|^{(n-k)q}} < \frac{1 - (1 - |z|^{-q})}{1 - |z|^{-q}} = \frac{|z|^{-q}}{1 - |z|^{-q}} = \frac{1}{|z|^q - 1},$$

so from $(*)$,

$$|p(z)| > |a_n||z|^n \left(1 - \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{1/p} \left(\frac{1}{|z|^q - 1} \right)^{1/q} \right). \quad (**)$$

So if $|z|^q - 1 \ge \left(\sum_{k=0}^{n-1} |a_k|^p / |a_n|^p \right)^{q/p}$ (notice that this also implies that $|z| > 1$) then $-1/(|z|^q - 1) \ge -\left(\sum_{k=0}^{n-1} |a_k|^p / |a_n|^p \right)^{-q/p}$ and by $(**)$,

$$|p(z)| > |a_n||z|^n \left(1 - \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{1/p} \left(\left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{-q/p} \right)^{1/q} \right) \ldots$$

Theorem 4 (continued 3)

Theorem 4. For any p and q such that $1/p + 1/q = 1$, $p > 1$, and $q > 1$, all zeros of polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ lie in

$$|z| < \left\{1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{q/p}\right\}^{1/q} \leq \left(1 + n^{q/p}\left(\max_{0\leq k\leq n-1} \frac{|a_k|}{|a_n|}\right)^q\right)^{1/q}.$$

Proof (continued). . . . or $|p(z)| > |a_n||z|^n(1-1) = 0$. So if

$$|z|^q - 1 \geq \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{q/p},$$

or if

$$|z| \geq \left(1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{q/p}\right)^{1/q},$$

then $p(z) \neq 0$, as claimed. □

Theorem 5. Joyal, Labelle, Rahman

Theorem 5. Joyal, Labelle, Rahman Generalization of Theorem 1. If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \leq \frac{1}{2}\left(1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right)$$

where $B = \max_{0\leq k<n-1} |a_k/a_n|$.

Proof. By the Triangle Inequality (and Exercise I.3.1)

$$|p(z)| \geq |a_n z^n + a_{n-1}z^{n-1}| - |a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_1 z + a_0|. \quad (1)$$

Next, also by the Triangle Inequality,

$$|a_n + a_{n-1}z^{n-1}| \geq |a_n||z|^n - |a_{n-1}||z|^{n-1} \quad (2)$$

and

$$|a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_1 z + a_0|$$
$$\leq |a_{n-2}||z|^{n-2} + |a_{n-3}||z|^{n-3} + \cdots + |a_1||z| + |a_0|$$

Theorem 5. Joyal, Labelle, Rahman (cont. 1)

Proof (continued).

$$\begin{aligned}
&\leq \max_{0\leq k<n-1} |a_k|(|z|^{n-2} + |z|^{n-3} + \cdots + |z| + 1) \\
&= B|a_n|(|z|^{n-2} + |z|^{n-3} + \cdots + |z| + 1) \\
&= B|a_n|\frac{|z|^{n-1} - 1}{|z| - 1} \text{ since } (|z|-1)(|z|^{n-2} + \cdots + |z| + 1) = |z|^{n-1} - 1 \\
&< B|a_n|\frac{|z|^{n-1}}{|z|-1}. \quad (3)
\end{aligned}$$

Combining (1), (2), and (3) gives

$$\begin{aligned}
|p(z)| &\geq |a_n z^n + a_{n-1}z^{n-1}| - |a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots a_1 z + a_0| \\
&> |a_n||z|^n - |a_{n-1}||z|^{n-1} - B|a_n|\frac{|z|^{n-1}}{|z|-1} \\
&= \frac{(|z|-1)(|a_n||z|^n - |a_{n-1}||z|^{n-1}) - B|a_n||z|^{n-1}}{|z|-1}
\end{aligned}$$

Theorem 5. Joyal, Labelle, Rahman (cont. 2)

Proof (continued).

$$\begin{aligned}
|p(z)| &> \frac{|a_n||z|^{n-1}}{|z|-1}\{(|z|-1)(|z| - |a_{n-1}/a_n|) - B\} \\
&= \frac{|a_n||z|^{n-1}}{|z|-1}\left\{|z|^2 + (-1 - |a_{n-1}/a_n|)|z| + |a_{n-1}/a_n| - B\right\}. \quad (4)
\end{aligned}$$

Now $|z|^2 + (-1 - |a_{n-1}/a_n|)|z| + |a_{n-1}/a_n| - B$ is a quadratic in $|z|$ so the graph (as a function of $|z|$) is a concave up parabola. The roots are

$$\begin{aligned}
|z| &= \frac{-(-1 - |a_{n-1}/a_n|) \pm \sqrt{(-1 - |a_{n-1}/a_n|)^2 - 4(1)(|a_{n-1}/a_n| - B)}}{2(1)} \\
&= \frac{1 + |a_{n-1}/a_n| \pm \sqrt{1 + 2|a_{n-1}/a_n| + |a_{n-1}/a_n|^2 - 4|a_{n-1}/a_n| + 4B)}}{2} \\
&= \frac{1}{2}\left\{1 + |a_{n-1}/a_n| \pm \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}.
\end{aligned}$$

Theorem 5. Joyal, Labelle, Rahman (cont. 3)

Proof (continued). Notice that $(1 - |a_{n-1}/a_n|)^2 + 4B \geq 0$, so the roots of the quadratic are real and the graph of the quadratic is concave up with one or two intercepts. In either case, for $|z|$ greater than the larger intercept, the quadratic is positive. That is, for

$$|z| > \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}$$

we have $\left\{|z|^2 + (-1 - |a_{n-1}/a_n|)|z| + |a_{n-1}/a_n| - B\right\} > 0$. Notice that

$$1+|a_{n-1}/a_n|+\sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B} \geq 1+|a_{n-1}/a_n|+1-|a_{n-1}/a_n| = 2$$

so that $|z| > \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}$ implies $|z| > 1$ and from (4) implies that $|p(z)| > 0$. So all zeros of p satisfy

$$|z| \leq \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\},$$

as claimed. □