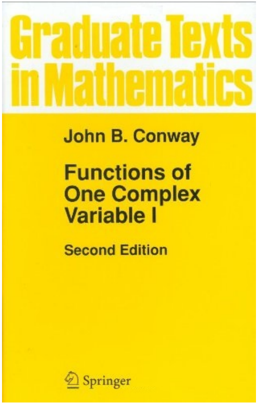
Complex Analysis

Chapter I. The Complex Number System

Supplement. Location of Zeros of Polynomials—Proofs of Theorems



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Cauchy's Location of Zeros Theorem, Category (1)

Theorem 1. Cauchy's Location of Zeros Theorem, Category (1).
If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \leq 1 + \max_{0 \leq k < n} |a_k/a_n| = \max_{0 \leq k < n} \frac{|a_n| + |a_k|}{|a_n|}.$$

Proof. Let $M = \max_{0 \leq k < n} |a_k/a_n|$. By the Triangle Inequality (and Exercise I.3.1),

$$\begin{aligned} |p(z)| &= \left|a_n z^n + (a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1})\right| \\ &\geq |a_n z^n| - |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}| \\ &\geq |a_n z^n| - (|a_0| + |a_1||z| + |a_2||z|^2 + \cdots + |a_{n-1}||z|^{n-1}). \qquad (*) \end{aligned}$$

Cauchy's Location of Zeros Theorem, Category (1)

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$$\begin{aligned} |p(z)| &= \left| a_n z^n + (a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}) \right| \\ &\geq |a_n z^n| - |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}| \\ &\geq |a_n z^n| - (|a_0| + |a_1||z| + |a_2||z|^2 + \cdots + |a_{n-1}||z|^{n-1}). \qquad (*) \end{aligned}$$

Cauchy's Location of Zeros Theorem, Category (1) (continued 1)

Proof (continued). Therefore, for $|z| > 1$ we have

$$\begin{aligned}
|p(z)| &\geq |a_n||z|^n - \left(\sum_{k=0}^{n-1} |a_k||z|^k\right) \text{ by } (*) \\
&= |a_n||z|^n \left(1 - \sum_{k=0}^{n-1} |a_k/a_n||z|^{k-n}\right) \\
&\geq |a_n||z|^n \left(1 - M\sum_{k=0}^{n-1} |z|^{k-n}\right) = |a_n||z|^n \left(1 - M\sum_{k=1}^{n} |z|^{-k}\right) \\
&\geq |a_n||z|^n \left(1 - M\sum_{k=1}^{\infty} |z|^{-k}\right)
\end{aligned}$$

...

Cauchy's Location of Zeros Theorem, Category (1) (continued 2)

Proof (continued). ...

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n \left(1 - M\frac{1/|z|}{1-1/|z|}\right) \text{ since } \sum_{k=1}^{\infty} |z|^{-k} \text{ is a} \\ &\quad \text{geometric series with ratio } 1/|z| < 1 \text{ and first term } 1/|z| \\ &= |a_n||z|^n \left(1 - \frac{M}{|z|-1}\right) = |a_n||z|^n \left(\frac{|z|-1-M}{|z|-1}\right). \end{aligned}$$

Hence, if $|z| > 1 + M$, then $|p(z)| > 0$ and so $p(z) \neq 0$. So all zeros of p in $|z| > 1$ must satisfy $|z| \leq 1 + M$; of course the zeros of p in $|z| \leq 1$ already lie in $|z| \leq 1 + M$. Therefore, all zeros of p lie in $|z| \leq 1 + M = 1 + \max_{0 \leq k < n} |a_k/a_n|$ (notice that if $M = \max_{0 \leq k < n} |a_k/a_n| = 0$ then $p(z) = a_n z^n$ and all zeros are at $z = 0$). □

Cauchy's Location of Zeros Theorem, Category (1) (continued 2)

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Theorem 2. Cauchy's Location of Zeros Theorem, Category (2)

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$$|a_n|x^n - (|a_{n-1}|x^{n-1} + |a_{n-2}|x^{n-2} + \cdots + |a_1|x + |a_0|) = 0.$$

Proof. As in the proof of Theorem 1, from the Triangle Inequality (and Exercise I.3.1) we have

$$|p(z)| \geq |a_n||z|^n - (|a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots |a_1||z| + |a_0|).$$

By Descartes' Rule of Signs, the equation

$$f(x) = |a_n|x^n - (|a_{n-1}|x^{n-1} + |a_{n-2}|x^{n-2} + \cdots |a_1|x + |a_0|) = 0$$

has exactly one real positive root r.

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Theorem 4

Theorem 4. Kuniyeda, Montel, and Tôya For any p and q such that $1/p + 1/q = 1$, $p > 1$, and $q > 1$, all zeros of polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ lie in

$$|z| < \left\{ 1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{q/p} \right\}^{1/q} \leq \left(1 + n^{q/p} \left(\max_{0 \leq k \leq n-1} \frac{|a_k|}{|a_n|} \right)^q \right)^{1/q}.$$

Proof. We have

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - (|a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots + |a_1||z| + |a_0|) \\ &= |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \text{ by Triangle Inequality and Exercise I.3.1} \\ &\geq |a_n||z|^n - \left(\sum_{k=0}^{n-1} |a_k|^p \right)^{1/p} \left(\sum_{k=0}^{n-1} |z|^{kq} \right)^{1/q} \text{ by Hölder's Inequality} \end{aligned}$$

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Theorem 4 (continued 1)

Proof (continued). ...

$$|p(z)| \geq |a_n||z|^n \left(1 - \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{1/p} \left(\sum_{k=0}^{n-1} \frac{1}{|z|^{(n-k)q}}\right)^{1/q}\right). \qquad (*)$$

If $|z| > 1$ then

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{1}{|z|^{(n-k)q}} &= \sum_{k=1}^{n} \frac{1}{|z|^{kq}} \text{ replacing } k \text{ with } n-k \\ &< \sum_{k=1}^{\infty} \frac{1}{|z|^{kq}} \text{ under the assumption that } |z| > 1 \\ &= \sum_{k=0}^{\infty} \frac{1}{|z|^{kq}} - 1 = \frac{1}{1-|z|^{-q}} - 1 \text{ since the series} \\ &\quad \text{is a geometric series with ratio } |z|^{-q} < 1 \end{aligned}$$

Theorem 4 (continued 2)

Proof (continued). ...

$$\sum_{k=0}^{n-1} \frac{1}{|z|^{(n-k)q}} < \frac{1-(1-|z|^{-q})}{1-|z|^{-q}} = \frac{|z|^{-q}}{1-|z|^{-q}} = \frac{1}{|z|^q - 1},$$

so from $(*)$,

$$|p(z)| > |a_n||z|^n \left(1 - \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{1/p} \left(\frac{1}{|z|^q - 1}\right)^{1/q}\right). \qquad (**)$$

So if $|z|^q - 1 \geq \left(\sum_{k=0}^{n-1} |a_k|^p/|a_n|^p\right)^{q/p}$ (notice that this also implies that $|z| > 1$) then $-1/(|z|^q - 1) \geq -\left(\sum_{k=0}^{n-1} |a_k|^p/|a_n|^p\right)^{-q/p}$ and by $(**)$,

$$|p(z)| > |a_n||z|^n \left(1 - \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{1/p} \left(\left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p}\right)^{-q/p}\right)^{1/q}\right) \ldots$$

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Theorem 4 (continued 3)

Theorem 4. For any p and q such that $1/p + 1/q = 1$, $p > 1$, and $q > 1$, all zeros of polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ lie in

$$|z| < \left\{ 1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{q/p} \right\}^{1/q} \leq \left(1 + n^{q/p} \left(\max_{0 \leq k \leq n-1} \frac{|a_k|}{|a_n|} \right)^q \right)^{1/q}.$$

Proof (continued). ...or $|p(z)| > |a_n||z|^n(1-1) = 0$. So if

$$|z|^q - 1 \geq \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{q/p},$$

or if

$$|z| \geq \left(1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{q/p} \right)^{1/q},$$

then $p(z) \neq 0$, as claimed. □

Theorem 5. Joyal, Labelle, Rahman

Theorem 5. Joyal, Labelle, Rahman Generalization of Theorem 1. If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \leq \frac{1}{2}\left(1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right)$$

where $B = \max_{0 \leq k < n-1} |a_k/a_n|$.

Proof. By the Triangle Inequality (and Exercise I.3.1)

$$|p(z)| \geq |a_n z^n + a_{n-1} z^{n-1}| - |a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \cdots + a_1 z + a_0|. \qquad (1)$$

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Next, also by the Triangle Inequality,

$$|a_n + a_{n-1} z^{n-1}| \geq |a_n||z|^n - |a_{n-1}||z|^{n-1} \quad (2)$$

and

$$|a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \cdots + a_1 z + a_0|$$
$$\leq |a_{n-2}||z|^{n-2} + |a_{n-3}||z|^{n-3} + \cdots + |a_1||z| + |a_0|$$

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Theorem 5. Joyal, Labelle, Rahman (cont. 1)

Proof (continued).

$$\begin{aligned}
&\le \max_{0\le k<n-1} |a_k|(|z|^{n-2} + |z|^{n-3} + \cdots + |z| + 1) \\
&= B|a_n|(|z|^{n-2} + |z|^{n-3} + \cdots + |z| + 1) \\
&= B|a_n|\frac{|z|^{n-1} - 1}{|z| - 1} \text{ since } (|z|-1)(|z|^{n-2} + \cdots + |z| + 1) = |z|^{n-1} - 1 \\
&< B|a_n|\frac{|z|^{n-1}}{|z| - 1}. \qquad (3)
\end{aligned}$$

Combining (1), (2), and (3) gives

$$\begin{aligned}
|p(z)| &\ge |a_n z^n + a_{n-1}z^{n-1}| - |a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots a_1 z + a_0| \\
&> |a_n||z|^n - |a_{n-1}||z|^{n-1} - B|a_n|\frac{|z|^{n-1}}{|z|-1} \\
&= \frac{(|z|-1)(|a_n||z|^n - |a_{n-1}||z|^{n-1}) - B|a_n||z|^{n-1}}{|z|-1}
\end{aligned}$$

Theorem 5. Joyal, Labelle, Rahman (cont. 1)

Proof (continued).

$$\begin{aligned} &\leq \max_{0\leq k<n-1} |a_k|(|z|^{n-2} + |z|^{n-3} + \cdots + |z| + 1) \\ &= B|a_n|(|z|^{n-2} + |z|^{n-3} + \cdots + |z| + 1) \\ &= B|a_n|\frac{|z|^{n-1} - 1}{|z| - 1} \text{ since } (|z|-1)(|z|^{n-2} + \cdots + |z| + 1) = |z|^{n-1} - 1 \\ &< B|a_n|\frac{|z|^{n-1}}{|z| - 1}. \qquad (3) \end{aligned}$$

Combining (1), (2), and (3) gives

$$\begin{aligned} |p(z)| &\geq |a_n z^n + a_{n-1}z^{n-1}| - |a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots a_1 z + a_0| \\ &> |a_n||z|^n - |a_{n-1}||z|^{n-1} - B|a_n|\frac{|z|^{n-1}}{|z|-1} \\ &= \frac{(|z|-1)(|a_n||z|^n - |a_{n-1}||z|^{n-1}) - B|a_n||z|^{n-1}}{|z|-1} \end{aligned}$$

Theorem 5. Joyal, Labelle, Rahman (cont. 2)

Proof (continued).

$$\begin{aligned} |p(z)| &> \frac{|a_n||z|^{n-1}}{|z|-1}\{(|z|-1)(|z|-|a_{n-1}/a_n|)-B\} \\ &= \frac{|a_n||z|^{n-1}}{|z|-1}\left\{|z|^2+(-1-|a_{n-1}/a_n|)|z|+|a_{n-1}/a_n|-B\right\}. \quad (4) \end{aligned}$$

Now $|z|^2+(-1-|a_{n-1}/a_n|)|z|+|a_{n-1}/a_n|-B$ is a quadratic in $|z|$ so the graph (as a function of $|z|$) is a concave up parabola. The roots are

$$\begin{aligned} |z| &= \frac{-(-1-|a_{n-1}/a_n|)\pm\sqrt{(-1-|a_{n-1}/a_n|)^2-4(1)(|a_{n-1}/a_n|-B)}}{2(1)} \\ &= \frac{1+|a_{n-1}/a_n|\pm\sqrt{1+2|a_{n-1}/a_n|+|a_{n-1}/a_n|^2-4|a_{n-1}/a_n|+4B)}}{2} \\ &= \frac{1}{2}\left\{1+|a_{n-1}/a_n|\pm\sqrt{(1-|a_{n-1}/a_n|)^2+4B}\right\}. \end{aligned}$$

Theorem 5. Joyal, Labelle, Rahman (cont. 2)

Proof (continued).

$$\begin{aligned} |p(z)| &> \frac{|a_n||z|^{n-1}}{|z|-1}\{(|z|-1)(|z|-|a_{n-1}/a_n|)-B\} \\ &= \frac{|a_n||z|^{n-1}}{|z|-1}\left\{|z|^2+(-1-|a_{n-1}/a_n|)|z|+|a_{n-1}/a_n|-B\right\}. \quad (4) \end{aligned}$$

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Theorem 5. Joyal, Labelle, Rahman (cont. 3)

Proof (continued). Notice that $(1 - |a_{n-1}/a_n|)^2 + 4B \geq 0$, so the roots of the quadratic are real and the graph of the quadratic is concave up with one or two intercepts. In either case, for $|z|$ greater than the larger intercept, the quadratic is positive. That is, for

$$|z| > \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}$$

we have $\left\{|z|^2 + (-1 - |a_{n-1}/a_n|)|z| + |a_{n-1}/a_n| - B\right\} > 0$. Notice that

$$1+|a_{n-1}/a_n|+\sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B} \geq 1+|a_{n-1}/a_n|+1-|a_{n-1}/a_n| = 2$$

so that $|z| > \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}$ implies $|z| > 1$ and from (4) implies that $|p(z)| > 0$.

Theorem 5. Joyal, Labelle, Rahman (cont. 3)

Proof (continued). Notice that $(1 - |a_{n-1}/a_n|)^2 + 4B \geq 0$, so the roots of the quadratic are real and the graph of the quadratic is concave up with one or two intercepts. In either case, for $|z|$ greater than the larger intercept, the quadratic is positive. That is, for

$$|z| > \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}$$

we have $\left\{|z|^2 + (-1 - |a_{n-1}/a_n|)|z| + |a_{n-1}/a_n| - B\right\} > 0$. Notice that

$$1+|a_{n-1}/a_n|+\sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B} \geq 1+|a_{n-1}/a_n|+1-|a_{n-1}/a_n| = 2$$

so that $|z| > \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\}$ implies $|z| > 1$ and from (4) implies that $|p(z)| > 0$. So all zeros of p satisfy

$$|z| \leq \frac{1}{2}\left\{1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right\},$$

as claimed. □

Theorem 5. Joyal, Labelle, Rahman (cont. 3)

Proof (continued). Notice that $(1-|a_{n-1}/a_n|)^2+4B \geq 0$, so the roots of the quadratic are real and the graph of the quadratic is concave up with one or two intercepts. In either case, for $|z|$ greater than the larger intercept, the quadratic is positive. That is, for

$$|z| > \frac{1}{2}\left\{1+|a_{n-1}/a_n|+\sqrt{(1-|a_{n-1}/a_n|)^2+4B}\right\}$$

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so that $|z| > \frac{1}{2}\left\{1+|a_{n-1}/a_n|+\sqrt{(1-|a_{n-1}/a_n|)^2+4B}\right\}$ implies $|z|>1$ and from (4) implies that $|p(z)|>0$. So all zeros of p satisfy

$$|z| \leq \frac{1}{2}\left\{1+|a_{n-1}/a_n|+\sqrt{(1-|a_{n-1}/a_n|)^2+4B}\right\},$$

as claimed. □