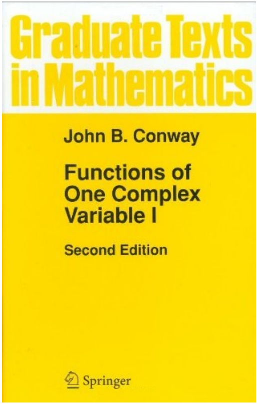
Complex Analysis

Chapter I. The Complex Number System

I.3. The Complex Plane—Proofs of Theorems



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$$\begin{aligned} |z + w|^2 &\leq |z|^2 + 2|z\overline{w}| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 \text{ since } |w| = |\overline{w}| \\ &= (|z| + |w|)^2. \end{aligned}$$

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Therefore $|z+w| \le |z| + |w|$. $\square$

Corollary

Corollary. For nonzero $z, w \in \mathbb{C}$, $|z+w| = |z| + |w|$ if and only if $z = tw$ for some $t \in \mathbb{R}$, $t \geq 0$.

Proof. First, if $z = tw$ then

$$\begin{aligned} |z+w| &= |tw+w| = |(t+1)w| = |t+1||w| \\ &= (t+1)|w| = t|w| + |w| \\ &= |tw| + |w| = |z| + |w|. \end{aligned}$$

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$$\frac{zw\overline{w}}{w} \geq 0 \text{ implies } \frac{z|w|^2}{w} = \frac{z}{w}|w|^2 = s \geq 0$$

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Cauchy Sequences Theorem

Theorem. A Cauchy sequence of complex numbers is convergent.

Proof. Let (z_n) be a Cauchy sequence of complex numbers and let $\text{Re}(z_n) = a_n$, $\text{Im}(z_n) = b_n$ for each $n \in \mathbb{N}$.

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Now $|a_m - a_n| = \sqrt{(a_m - a_n)^2} \leq \sqrt{(a_m - a_n)^2 + (b_m - b_n)^2} = |z_m - z_n|$, so if $m, n \geq N$ then $|a_m - a_n| \leq |z_m - z_n| < \varepsilon$, and (a_n) is a Cauchy sequence of *real* numbers.

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