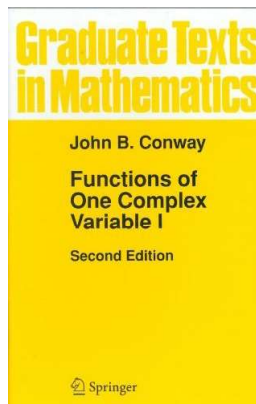


Complex Analysis

Chapter II. Metric Spaces and the Topology of \mathbb{C}

II.1. Definitions and Examples of Metric Spaces—Proofs of Theorems



Theorem II.1.13(c1)

Theorem II.1.13(c1). Let X be a metric space and $A \subset X$. Then $\text{int}(A) = X \setminus (X \setminus A)^-$.

Proof. Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^-$. Hence $B(x; \varepsilon) \cap (X \setminus A)^- = \emptyset$, and so $B(x; \varepsilon) \cap (X \setminus A) = \emptyset$. Therefore $B(x; \varepsilon) \subset A$. Since $B(x; \varepsilon)$ is open and a subset of set A , then $x \in \text{int}(A)$. So $X \setminus (X \setminus A)^- \subset \text{int}(A)$.

Let $x \in \text{int}(A)$. Then since $\text{int}(A)$ is open (it is the union of a collection of open sets; use Theorem 1.9(c)). So (by definition) there is $\varepsilon > 0$ such that $B(x; \varepsilon) \subset \text{int}(A) \subset A$. Now $X \setminus B(x; \varepsilon)$ is closed and $X \setminus A \subset X \setminus B(x; \varepsilon)$, so $(X \setminus A)^- \subset X \setminus B(x; \varepsilon)$ (the set on the right-hand side is closed by definition) and $x \notin (X \setminus A)^-$. Therefore $x \in X \setminus (X \setminus A)^-$ and so $\text{int}(A) \subset X \setminus (X \setminus A)^-$. Hence, $\text{int}(A) = X \setminus (X \setminus A)^-$. \square

Theorem II.1.13(f)

Theorem II.1.13(f). Let X be a metric space and $A \subset X$. Then $x_0 \in A^-$ if and only if for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \emptyset$.

Proof. Let $x_0 \in A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \notin \text{int}(X \setminus A)$. By part (e), for every $\varepsilon > 0$ the ball $B(x_0; \varepsilon)$ is not a subset of $X \setminus A$. That is, there is $y \in B(x_0; \varepsilon) \cap A$ (for any $\varepsilon > 0$ there is some such y). Therefore, for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \emptyset$.

Now suppose $x_0 \notin A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \in \text{int}(X \setminus A)$ and, by part (e), there is $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset X \setminus A$. But then $B(x_0; \varepsilon) \cap A = \emptyset$ and x_0 does not satisfy the condition $B(x_0; \varepsilon) \cap A \neq \emptyset$ (here, we have proven the contrapositive of the 'only if' part). \square