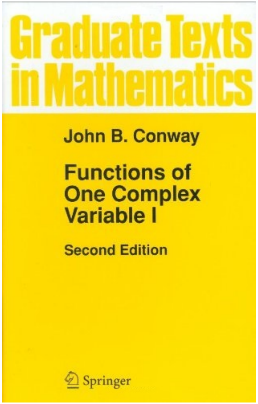
Complex Analysis

Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.1. Definitions and Examples of Metric Spaces—Proofs of Theorems



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Proof. Let $x \in X \setminus (X \setminus A)^{-}$. Well, $(X \setminus A)^{-}$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^{-}$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^{-}$.

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Proof. Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset X \setminus (X \setminus A)^-$. Hence $B(x;\varepsilon) \cap (X \setminus A)^- = \varnothing$, and so $B(x;\varepsilon) \cap (X \setminus A) = \varnothing$. Therefore $B(x;\varepsilon) \subset A$. Since $B(x;\varepsilon)$ is open and a subset of set A, then $x \in \text{int}(A)$. So $X \setminus (X \setminus A)^- \subset \text{int}(A)$.

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Let $x \in \text{int}(A)$. Then since $\text{int}(A)$ is open (it is the union of a collection of open sets; use Theorem 1.9(c)). So (by definition) there is $\varepsilon > 0$ such that $B(x;\varepsilon) \subset \text{int}(A) \subset A$.

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Theorem II.1.13(f)

Theorem II.1.13(f). Let X be a metric space and $A \subset X$. Then $x_0 \in A^-$ if and only if for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \varnothing$.

Proof. Let $x_0 \in A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \notin \text{int}(X \setminus A)$. By part (e), for every $\varepsilon > 0$ the ball $B(x_0; \varepsilon)$ is not a subset of $X \setminus A$. That is, there is $y \in B(x_0; \varepsilon) \cap A$ (for any $\varepsilon > 0$ there is some such y). Therefore, for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \varnothing$.

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Now suppose $x_0 \notin A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \in \text{int}(X \setminus A)$ and, by part (e), there is $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset X \setminus A$. But then $B(x_0; \varepsilon) \cap A = \varnothing$ and x_0 does not satisfy the condition $B(x_0; \varnothing) \cap A \neq \varnothing$ (here, we have proven the contrapositive of the 'only if' part). □

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