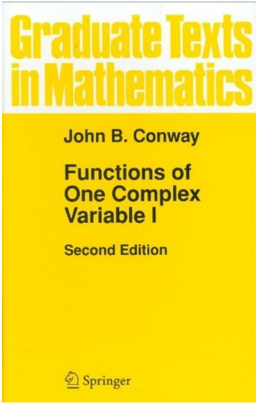
Complex Analysis

Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.2. Connectedness—Proofs of Theorems



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Theorem II.2.3. An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a, b \in G$ there is a polygon from a to b lying entirely in G.

Proof. Suppose that for open set G, for any two points $a, b \in G$ there is a polygon in G from a to b. ASSUME that G is not connected. Then there are sets A and B such that A and B are both open and closed, $G \subset A \cup B$, $A \cap B = \varnothing$, and $A \neq \varnothing \neq B$. Let $a \in A$ and $b \in B$ with polygon P from a to b such that $P \subset G$. Now there must be at least one segment in P with one endpoint in A and the other endpoint in B (or else the polygon cannot connect a point in A to a point in B; remember that $A \cap B = \varnothing$). Say the segment is $[z_k, w_k]$.

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$$S = \{s \in [0,1] \mid sw_k + (1-s)z_k \in A\}$$

$$T = \{t \in [0,1] \mid tw_k + (1-t)z_k \in B\}.$$

Then $S \cap T = \varnothing$ (since $A \cap B = \varnothing$), $S \cup T = [0,1]$ (since $[z_k, w_k] \subset G \subset A \cup B$), $0 \in S$, and $1 \in T$.

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Theorem II.2.3 (continued 1)

We need the following exercise:

Exercise II.2.2. Sets S and T are both open.

Proof. Let $s' \in S$. Then $a' = s'w_k + (1 - s')z_k \in A$. Since A is open, there is $\varepsilon > 0$ such that $B(a'; \varepsilon) \subset A$.

Theorem II.2.3 (continued 1)

We need the following exercise:

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Proof. Let $s' \in S$. Then $a' = s'w_k + (1-s')z_k \in A$. Since A is open, there is $\varepsilon > 0$ such that $B(a';\varepsilon) \subset A$. Then for all

$$s \in \big(s' - \varepsilon/|z_k - w_k|, s' + \varepsilon/|z_k - w_k|\big) \cap [0,1],$$

for $a = sw_k + (1-s)z_k$ we have

$$\begin{aligned} d(a,a') &= |a - a'| = |(sw_k + (1-s)z_k) - (s'w_k + (1-s')z_k)| \\ &= |(s-s')w_k + (s'-s)z_k| = |s'-s||z_k - w_k| \\ &< (\varepsilon/|z_k - w_k|)|z_k - w_k| = \varepsilon. \end{aligned}$$

So

$$\big(s' - \varepsilon/|z_k - w_k|, s' + \varepsilon/|z_k - w_k|\big) \cap [0,1] \subset S$$

and therefore S is open (relative to $[0,1]$). Similarly, set T is open. □

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Theorem II.2.3 (continued 2)

Theorem II.2.3. An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a, b \in G$ there is a polygon from a to b lying entirely in G.

Proof (continued). But then sets S and T are both open and closed sets with respect to $[0, 1]$. That is, S, T form a separation of $[0, 1]$. But this implies that $[0, 1]$ is not connected, a CONTRADICTION to Proposition II.2.2. Therefore the assumption that G is not connected is false, and G is connected.

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Now suppose that G is open and connected. Fix point $a \in G$. Define

$$A = \{b \in G \mid \text{ there is a polygon from } a \text{ to } b\}.$$

We want to show that $A = G$. We do so by showing that A is both open and closed in the metric space. Since G is open and connected, then either $A = \varnothing$ (which is not the case since $a \in A$) or $A = G$.

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Theorem II.2.3 (continued 3)

Theorem II.2.3. An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a, b \in G$ there is a polygon from a to b lying entirely in G.

Proof (continued). Claim 1. The set $A = \{b \in G \mid \text{ there is a polygon from } a \text{ to } b\}$ is open.

Let $b \in A$ and let $P = [a, z_1, z_2, \ldots, b]$ be a polygon in G. G is open so there exists $\varepsilon > 0$ such that $B(b; \varepsilon) \subset G$. So for all $z \in B(b; \varepsilon)$, the line segment $[b, z] \subset B(b; \varepsilon)$. So the polygon $P \cup [b, z] \subset G$ and this polygon goes from a to z. So $z \in A$. Therefore, $B(b; \varepsilon) \subset A$ and hence A is open.

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Theorem II.2.3. An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a, b \in G$ there is a polygon from a to b lying entirely in G.

Proof (continued). Claim 2. The set
$A = \{b \in G \mid \text{ there is a polygon from } a \text{ to } b\}$ is closed (in G).

If $G = A$, we are done. So consider, without loss of generality, $G \setminus A \neq \varnothing$ and let $z \in G \setminus A$. Let $\varepsilon > 0$ be such that $B(z;\varepsilon) \subset G$ (this can be done since G is open). ASSUME that $b \in A \cap B(z;\varepsilon)$.

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So set A is both open and closed, and the result follows as described above. □

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Lemma II.2.6

Lemma II.2.6. Let $x_0 \in X$ and let $\{D_j \mid j \in J\}$ be a collection of connected subsets of X such that $x_0 \in D_j$ for all $j \in J$. Then $D = \cup_{j \in J} D_j$ is connected.

Proof. Let A be a subset of metric space (D, d) which is both open and closed and suppose $A \neq \varnothing$. Then $A \cap D_j$ is open in (D_j, d) for all $j \in J$ (by the definition of 'open relative to D_j'). Also, $A \cap D_j$ is closed in (D_j, d) for all $j \in J$ (by the definition of 'closed relative to D_j'; these claims of open and closed are justified rigorously in Exercises II.1.8 and II.1.9). Since D_j is connected, then either $A \cap D_j = \varnothing$ or $A \cap D_j = D_j$ (since $A \cap D_j$ is both open and closed).

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Theorem II.2.7

Theorem II.2.7. Let (X, d) be a metric space. Then:

(a) each $x_0 \in X$ is contained in some component of X, and

(b) distinct components of X are disjoint.

Proof of (a). Let $\mathcal{D}$ be the collection of connected subsets of X which contain point x_0. By definition $\{x_0\} \in \mathcal{D}$, so $\mathcal{D} \neq \varnothing$. By Lemma II.2.6, $C = \cup_{D \in \mathcal{D}} D$ is connected and $x_0 \in C$. Next, if D is a connected set and $C \subset D$, then $x_0 \in D$ and $D \in \mathcal{D}$. But then $D \subset C$ and so $C = D$. So C is a maximal connected set containing x_0. That is, C is a component of space X.

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Proof of (b). Suppose C_1 and C_2 are components with $C_1 \neq C_2$. ASSUME $x_0 \in C_1 \cap C_2$. Then by Lemma II.2.6, $C_1 \cup C_2$ is connected. But if C_1 and C_2 are components, then they are maximal connected subsets and so $C_1 = C_1 \cup C_2 = C_2$, a CONTRADICTION to the fact that $C_1 \neq C_2$. So the assumption that there is $x_0 \in C_1 \cap C_2$ is false and different components are disjoint. □

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Theorem II.2.9

Theorem II.2.9. Let G be open in $\mathbb{C}$. Then the components of G are open and there are only a countable number of them.

Proof. Let C be a component of G and let $x_0 \in C$. Since G is open, there exists $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset G$. So $x_0 \in B(x_0; \varepsilon) \cap C$ and by Lemma II.2.6, $B(x_0; \varepsilon) \cup C$ is connected. But C is a component, so it is a maximal connected set and hence $B(x_0; \varepsilon) \cup C = C$. That is, $B(x_0; \varepsilon) \subset C$ and so C is open.

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For countable, let $S = \{a + ib \mid a, b \in \mathbb{Q} \text{ and } a + ib \in G\}$. Then S is countable and each component of G contains a point of S with different components containing different points since the components are disjoint (by Theorem II.2.7(b)). So the number of components is countable. □

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Proof. Let C be a component of G and let $x_0 \in C$. Since G is open, there exists $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset G$. So $x_0 \in B(x_0; \varepsilon) \cap C$ and by Lemma II.2.6, $B(x_0; \varepsilon) \cup C$ is connected. But C is a component, so it is a maximal connected set and hence $B(x_0; \varepsilon) \cup C = C$. That is, $B(x_0; \varepsilon) \subset C$ and so C is open.

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