Complex Analysis

Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.3. Sequences and Completeness—Proofs of Theorems

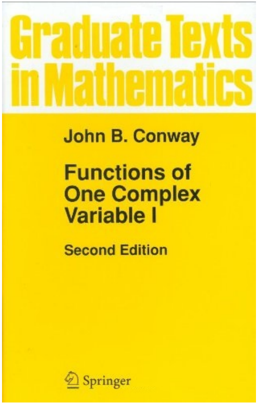

Table of contents

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Proof. Suppose F is closed and $x = \lim x_n$ where each $x_n \in F$. So for all $\varepsilon > 0$ we have $x_n \in B(x;\varepsilon)$ for some $x_n \in \{x_n\}$. Then $B(x;\varepsilon) \cap F \neq \varnothing$. So by Proposition 1.13(f), $x \in F^-$. Since F is closed, $F = F^-$ and $x \in F$.

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Next, suppose F is not closed. Then there is some $x_0 \in F^-$ where $x_0 \notin F$. Again by Proposition 1.13(f), for every $\varepsilon > 0$ we have $B(x_0; \varepsilon) \cap F \neq \varnothing$. In particular, for each $n \in \mathbb{N}$, with $\varepsilon = 1/n$, we can choose some $x \in B(x_0; 1/n) \cap F$ and denote it as x_n. Then the sequence $\{x_n\}$ thus created, converges to x_0. But then $\{x_n\} \to x_0 \notin F$ and so the sequence condition does not hold. □

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Cantor's Theorem

Cantor's Theorem. A metric space (X, d) is complete if and only if for any sequence $\{F_n\}$ of nonempty closed sets with $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $\text{diam}(F_n) \to 0$, then the set $\cap_{n=1}^{\infty} F_n$ consists of a single point.

Proof. Suppose (X, d) is complete and let $\{F_n\}$ be a sequence of closed sets having the properties: (i) $F_1 \supset F_2 \supset \cdots$ and (ii) $\text{diam}(F_n) \to 0$. We now show $\cap F_n$ consists of a single point. For $n \in \mathbb{N}$, let $x_n \in F_n$. If $n, m \geq N$ then $x_n, x_m \in F_N$ and so $d(x_n, x_m) \leq \text{diam}(F_N)$. Since $\text{diam}(F_n) \to 0$, for any given $\varepsilon > 0$, N can be chosen sufficiently large so that $\text{diam}(F_N) < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

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Proof (continued). Now suppose (X, d) satisfies the condition on nested closed sets. Let $\{x_n\}$ be a Cauchy sequence in X. Define $F_n = \{x_n, x_{n+1}, \ldots\}^-$; then $F_1 \supset F_2 \supset F_3 \supset \cdots$. Let $\varepsilon > 0$. Choose N such that for all $m, n \geq N$ we have $d(x_n, x_m) < \varepsilon$. So for $n \geq N$ we have $\text{diam}\{x_n, x_{n+1}, \ldots\} \leq \varepsilon$. So, by Exercise II.3.3, for $n \geq N$,

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So $\text{diam}(F_n) \to 0$. Then by hypothesis there is $x_0 \in X$ with $\{x_0\} = F_1 \cap F_2 \cap \cdots$. In particular, $x_0 \in F_n$ for all $n \in \mathbb{N}$ and so $d(x_0, x_n) \leq \text{diam}(F_n)$. But $\text{diam}(F_n) \to 0$, so $x_n \to x_0$ and sequence $\{x_n\}$ converges to x_0. So (X, d) is complete, as claimed. □

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Proposition II.3.8

Proposition II.3.8. Let (X, d) be a complete metric space and let $Y \subset X$. Then (Y, d) is a complete metric space if and only if Y is closed in X.

Proof. Suppose (Y, d) is complete. Let x_0 be a limit point of Y. Then there is a sequence $\{y_n\}$ of distinct points in Y such that $x_0 = \lim(y_n)$. By Exercise II.3.5, $\{y_n\}$ is Cauchy and so converges to some $y_0 \in Y$, since (Y, d) is complete. Since limits of sequences are unique, then $y_0 = x_0$ and $x_0 \in Y$. So Y contains its limit points and by Proposition II.3.4(a) Y is closed. The converse is Exercise II.3.2. □

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