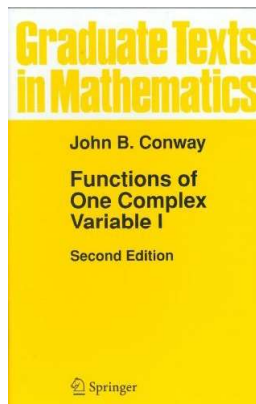


Complex Analysis

Chapter II. Metric Spaces and the Topology of \mathbb{C} II.4. Compactness—Proofs of Theorems



Proposition II.4.3

Proposition II.4.3. Let K be a compact subset of X . Then

- (a) K is closed, and
- (b) if F is closed and $F \subset K$ then F is compact.

Proof. (a) Let $x_0 \in K^-$. We show that $x_0 \in K$ and $K = K^-$ (so K is closed). Let $\varepsilon > 0$. Then

$$B(x_0; \varepsilon) \cap K \neq \emptyset \quad (1)$$

by Theorem 1.13(f). For each $n \in \mathbb{N}$, define (open) $G_n = X \setminus B(x_0; 1/n)^-$. ASSUME $x_0 \notin K$. Then each G_n is open and $K \subset \bigcup_{n=1}^{\infty} G_n = X \setminus \{x_0\}$. Since K is compact (by hypothesis), then $K \subset \bigcup_{n=1}^m G_n$ for some $m \in \mathbb{N}$ (with possible relabeling of the G_n 's) where $G_1 \subset G_2 \subset \dots \subset G_m$. Then $K \subset G_m = X \setminus B(x_0, 1/m)^-$. This implies that $B(x_0, 1/m)^- \cap K = \emptyset$, CONTRADICTING (1). So $x_0 \in K$, $K = K^-$ and K is closed.

Proposition II.4.3

Proposition II.4.3. Let K be a compact subset of X . Then

- (a) K is closed, and
- (b) if F is closed and $F \subset K$ then F is compact.

Proof. (b) Let \mathcal{G} be an open cover of F . Since F is closed, then $X \setminus F$ is open. So $\mathcal{G} \cup \{X \setminus F\}$ is an open cover of K . Since K is compact, there are G_1, G_2, \dots, G_n in \mathcal{G} such that $K \subset G_1 \cup G_2 \cup \dots \cup G_n \cup (X \setminus F)$. Since $F \subset K$, then $F \subset G_1 \cup G_2 \cup \dots \cup G_n$ and so F is compact. \square

Proposition II.4.4

Proposition II.4.4. A set $K \subset X$ is compact if and only if every collection \mathcal{F} of closed subsets of K with the finite intersection property satisfies $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

Proof. Suppose K is compact and \mathcal{F} is a collection of closed subsets of K having the finite intersection property. ASSUME $\bigcap_{F \in \mathcal{F}} F = \emptyset$ and let $\mathcal{G} = \bigcup_{F \in \mathcal{F}} (X \setminus F)$. Then

$$\begin{aligned} \bigcup_{F \in \mathcal{F}} (X \setminus F) &= X \setminus \bigcap_{F \in \mathcal{F}} F \text{ by DeMorgan's Laws} \\ &= X \text{ by assumption.} \end{aligned}$$

So \mathcal{G} is an open cover of K . Thus, there are $F_1, F_2, \dots, F_n \in \mathcal{F}$ such that $K \subset \bigcup_{k=1}^n (X \setminus F_k) = X \setminus \bigcap_{k=1}^n F_k$ by DeMorgan. But then $\bigcap_{k=1}^n F_k \subset X \setminus K$. Since for each k , we have $F_k \subset K$ by definition of \mathcal{F} , it must be that $\bigcap_{k=1}^n F_k = \emptyset$ (the only subset of K which is a subset of $X \setminus K$ is \emptyset). But this CONTRADICTS the finite intersection property. So the assumption that $\bigcap_{F \in \mathcal{F}} F = \emptyset$ is false and hence $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

Proposition II.4.4 (continued)

Proposition II.4.4. A set $K \subset X$ is compact if and only if every collection \mathcal{F} of closed subsets of K with the finite intersection property satisfies $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

Proof (continued). Now suppose every collection \mathcal{F} of closed subsets of K with the finite intersection property satisfies $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. ASSUME K is not compact. Let \mathcal{G} be an open cover of K with no finite subcover and define $\mathcal{F} = \{K \setminus G \mid G \in \mathcal{G}\}$. Then \mathcal{F} consists of sets closed in K and for any $F_1, F_2, \dots, F_n \in \mathcal{F}$ we have

$$F_1 \cap F_2 \cap \dots \cap F_n = (K \setminus G_1) \cap (K \setminus G_2) \cap \dots \cap (K \setminus G_n) = K \setminus \bigcup_{k=1}^n G_k.$$

Since K is not compact, then $\bigcup_{k=1}^n G_k$ does not cover K and hence $K \setminus \bigcup_{k=1}^n G_k \neq \emptyset$. So \mathcal{F} satisfies the finite intersection property. However, $\bigcap_{F \in \mathcal{F}} F = K \setminus \bigcup_{G \in \mathcal{G}} G = \emptyset$ since \mathcal{G} is an open cover of K , a CONTRADICTION. So the assumption that K is not compact is false and K is compact. \square

Corollary II.4.5

Corollary II.4.5. Every compact metric space is complete.

Proof. We use Cantor's Theorem (Theorem II.3.7). Let $\{F_n\}$ be a sequence of non-empty closed sets with $F_1 \supset F_2 \supset \dots$ and $\text{diam}(F_n) \rightarrow 0$. Since the F 's are nested, any finite collection satisfies $F_{n_1} \cap F_{n_2} \cap \dots \cap F_{n_k} \neq \emptyset$ (where $n_1 < n_2 < \dots < n_k$), so $\{F_n\}$ has the finite intersection property. By Proposition II.4.4, since the metric space is complete, $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$. So there is some $x \in \bigcap_{n \in \mathbb{N}} F_n$. Since $\text{diam}(F_n) \rightarrow 0$, there is a unique such x (as argued in the proof of Cantor's Theorem) and so by Cantor's Theorem, the metric space is complete. \square

Corollary II.4.6

Corollary II.4.6. If X is a compact set in a metric space, then every infinite set has a limit point in X .

Proof. Let S be an infinite subset of X . ASSUME S has no limit points in X . Let $\{a_1, a_2, \dots\}$ be a sequence of distinct points in S . Define $F_n = \{a_n, a_{n+1}, \dots\}$. Then since S has no limit points, set F_n has no limit points for all $n \in \mathbb{N}$. So F_n contains all of its limit points (it has none!) and F_n is closed for all $n \in \mathbb{N}$ by Proposition II.3.4(a). As in the proof of Corollary II.4.5, since the F_n are nested, they satisfy the finite intersection property. But the a_n 's are distinct and so $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$. By Proposition II.4.4, this implies that X is not compact, a CONTRADICTION. So the assumption that S has no limit points in X is false. \square

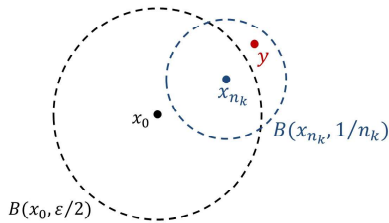
Lemma II.4.8

Lemma II.4.8. Lebesgue's Covering Lemma.

If (X, d) is sequentially compact and \mathcal{G} is an open cover of X then there is an $\varepsilon > 0$ such that if $x \in X$, there is a set $G \in \mathcal{G}$ with $B(x; \varepsilon) \subset G$.

Proof. Suppose \mathcal{G} is an open cover of X . ASSUME there is no such ε . Then for each $n \in \mathbb{N}$ there is a point $x_n \in X$ such that $B(x_n; 1/n)$ is not contained in any set $G \in \mathcal{G}$. Consider the sequence $\{x_n\}$. Since X is sequentially compact, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow x_0$ for some $x_0 \in X$. Since \mathcal{G} is a covering of X then $x_0 \in G_0$ for some $G_0 \in \mathcal{G}$ and since G_0 is open, for some $\varepsilon > 0$ we have $B(x_0; \varepsilon) \subset G_0$. Let $N \in \mathbb{N}$ be such that $d(x_0; x_{n_k}) < \varepsilon/2$ for all $n_k \geq N$. Next, let $n_k > \max\{N, 2/\varepsilon\}$ and let $y \in B(x_{n_k}; 1/n_k)$ (see the image below).

Lemma II.4.8 (continued)



Then

$$\begin{aligned} d(x_0, y) &\leq d(x_0, x_{n_k}) + d(x_{n_k}, y) \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + 1/n_k \text{ by the choices of } n_k \text{ and } y \\ &< \varepsilon. \end{aligned}$$

But then $B(x_{n_k}; 1/n_k) \subset B(x_0; \varepsilon) \subset G$. However, we originally chose x_n such that $B(x_n; 1/n)$ is not contained in any $G \in \mathcal{G}$, a CONTRADICTION for $n = n_k$. So the assumption that there is no $\varepsilon > 0$ as described is false, and the result follows. \square

Proposition II.4.9

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (a) X is compact,
- (b) every infinite subset of X has a limit point,
- (c) X is sequentially compact, and
- (d) X is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \dots, x_n \in X$ such that $X = \cup_{k=1}^n B(x_k; \varepsilon)$. This property is called *total boundedness*.

Proof. (a) implies (b): This is Corollary II.4.6.

(b) implies (c): Let $\{x_n\}$ be a sequence in X . Without loss of generality, the x_n are distinct (if an element is repeated an infinite number of times then it is easy to produce a convergent subsequence; finite repetitions have no effect). By hypothesis, the set $\{x_1, x_2, \dots\}$ has a limit point, say $x_0 \in X$. So some $x_{n_1} \in B(x_0; 1)$; some $x_{n_2} \in B(x_0, 1/2)$ where $n_2 > n_1$; and in general $x_{n_k} \in B(x_0; 1/k)$ where $n_k > n_{k-1} > \dots > n_2 > n_1$.

Proposition II.4.9 (continued 1)

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (c) X is sequentially compact, and
- (d) X is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \dots, x_n \in X$ such that $X = \cup_{k=1}^n B(x_k; \varepsilon)$. This property is called *total boundedness*.

Proof (continued). Then subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has limit $x_0 \in X$. That is, every sequence has a convergent subsequence and X is sequentially compact.

(c) implies (d): Let $\{x_n\}$ be a Cauchy sequence. By hypothesis, $\{x_n\}$ has a convergent subsequence which converges, say, to x_0 . Then $\{x_n\} \rightarrow x_0$ (by Exercise II.3.8) and so X is complete. Now let $\varepsilon > 0$. Fix $x_1 \in X$. If $X = B(x_1; \varepsilon)$ then we conclude (d) with $n = 1$. Otherwise choose $x_2 \in X \setminus B(x_1; \varepsilon)$. If $X = B(x_1; \varepsilon) \cup B(x_2; \varepsilon)$ then we conclude (d) with $n = 2$. Continue in this way and construct x_3, x_4, \dots . If the process terminates at some n , then we conclude (d).

Proposition II.4.9 (continued 2)

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (c) X is sequentially compact, and
- (d) X is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \dots, x_n \in X$ such that $X = \cup_{k=1}^n B(x_k; \varepsilon)$. This property is called *total boundedness*.

Proof (continued.) If the process does not terminate then we have a sequence $\{x_n\}$ where for any $n \neq m$ we have $d(x_n, x_m) \geq \varepsilon$. But then sequence $\{x_n\}$ can have no convergent subsequence, violating the hypothesis. So the process must terminate and we have $X = \cup_{k=1}^m B(x_k; \varepsilon)$ for some $\{x_1, x_2, \dots, x_n\}$.

(d) implies (c): Let $\{x_n\}$ be a sequence of distinct points (as in the proof of (b) implies (c) we may assume WLOG that the points are distinct). With $\varepsilon = 1$, we can write X as a finite union of balls of radius 1. Since $\{x_n\}$ is an infinite set, there is some $y_1 \in X$ such that $B(y_1; 1)$ contains an infinite number of the x_n 's, say $\{x_n^{(1)}\}_{n=1}^\infty$.

Proposition II.4.9 (continued 3)

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (c) X is sequentially compact, and
- (d) X is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \dots, x_n \in X$ such that $X = \bigcup_{k=1}^n B(x_k; \varepsilon)$. This property is called *total boundedness*.

Proof (continued.) Similarly with $\varepsilon = 1/2$, there is $y_2 \in X$ and subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ with $\{x_n^{(2)}\} \subset B(y_2; 1/2)$. In general, for each $k \in \mathbb{N}$, $k \geq 2$, there is $y_k \in X$ and subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ with $\{x_n^{(k)}\} \subset B(y_k, 1/k)$. Define $F_k = \{x_n^{(k)}\}^-$. Then $\text{diam}(F_k) \leq 2/k$ and $F_1 \supset F_2 \supset \dots$. Since X is complete by hypothesis, then by Theorem II.3.7, $\bigcap_{k=1}^{\infty} F_k = \{x_0\}$ for some $x_0 \in X$. Consider the subsequence of $\{x_n\}$ defined as $\{x_k^{(k)}\}$ (by the recursive way the subsequences are constructed, we are insured that this is in fact a subsequence).

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Proposition II.4.9 (continued 4)

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (a) X is compact,
- (c) X is sequentially compact.

Proof (continued.) Notice that

$$\begin{aligned} d(x_0, x_k^{(k)}) &\leq \text{diam}(F_k) \text{ (since } x_0, x_k^{(k)} \in F_k) \\ &\leq 2/k \end{aligned}$$

for all $k \in \mathbb{N}$. So $x_k^{(k)} \rightarrow x_0 \in X$. So $\{x_n\}$ has a convergent subsequence and X is sequentially compact.

(c) implies (a): Let \mathcal{G} be an open cover of X . Since X is hypothesized to be sequentially compact, then by Lebesgue's Covering Lemma (Lemma II.4.8), there is some $\varepsilon > 0$ such that for every $x \in X$, there is a $G \in \mathcal{G}$ where $B(x; \varepsilon) \subset G$.

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Proposition II.4.9 (continued 5)

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (a) X is compact,
- (c) X is sequentially compact.

Proof (continued.) From above, we have that (c) implies (d), so there are points $x_1, x_2, \dots, x_n \in X$ such that $X = \bigcup_{k=1}^n B(x_k; \varepsilon)$. Again, by Lebesgue's Covering Lemma, for $k = 1, 2, \dots, n$ we have some $G_k \in \mathcal{G}$ such that $B(x_k; \varepsilon) \subset G_k$. Then $X = \bigcup_{k=1}^n G_k$ and $\{G_1, G_2, \dots, G_n\}$ is a finite subcover of \mathcal{G} . So X is compact and (a) follows. \square

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Heine-Borel Theorem

Theorem II.4.10. Heine-Borel Theorem.

A subset K of \mathbb{R}^n ($n \geq 1$) is compact if and only if K is closed and bounded.

Proof. Suppose K is compact. Then K is closed by Proposition II.4.3(a). By Theorem II.4.9(d), X is totally bounded. So there is $\varepsilon > 0$ and x_1, x_2, \dots, x_n such that $X = \bigcup_{k=1}^n B(x_k; \varepsilon)$. We then have that $d(x_1, y) < 2n\varepsilon$ for all $y \in X$ and so X is bounded.

Now suppose K is closed and bounded. Since $K \subset \mathbb{R}^n$ is bounded, then for some a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n we have $K \subset [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] = F$. Now F is closed (consider $\mathbb{R}^n \setminus F$) and \mathbb{R}^n is complete (here is where completeness is used—showing \mathbb{R}^n is complete based on the completeness of \mathbb{R} is similar to the proof that \mathbb{C} is complete in Proposition II.3.6). So F is complete by Proposition II.3.8. By Lemma, F is totally bounded. So by Proposition II.4.9(d), set F is compact. Now by Proposition II.4.3(b), since set K is closed, set K is compact. \square

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