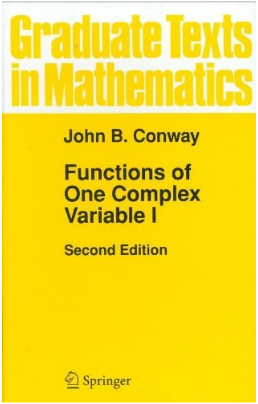
# Complex Analysis

## Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

### II.4. Compactness—Proofs of Theorems



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# Proposition II.4.3

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(a) $K$ is closed, and

(b) if $F$ is closed and $F \subset K$ then $F$ is compact.

**Proof. (a)** Let $x_0 \in K^-$. We show that $x_0 \in K$ and $K = K^-$ (so $K$ is closed). Let $\varepsilon > 0$. Then

$$B(x_0; \varepsilon) \cap K \neq \varnothing \qquad (1)$$

by Theorem 1.13(f). For each $n \in \mathbb{N}$, define (open) $G_n = X \setminus B(x_0; 1/n)^-$.

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by Theorem 1.13(f). For each $n \in \mathbb{N}$, define (open) $G_n = X \setminus B(x_0;1/n)^-$. ASSUME $x_0 \notin K$. Then each $G_n$ is open and $K \subset \cup_{n=1}^{\infty} G_n = X \setminus \{x_0\}$. Since $K$ is compact (by hypothesis), then $K \subset \cup_{n=1}^{m} G_n$ for some $m \in \mathbb{N}$ (with possible relabeling of the $G_n$'s) where $G_1 \subset G_2 \subset \cdots \subset G_m$. Then $K \subset G_m = X \setminus B(x_0, 1/m)^-$. This implies that $B(x_0;1/m)^- \cap K = \varnothing$, CONTRADICTING (1). So $x_0 \in K$, $K = K^-$ and $K$ is closed.

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(a) $K$ is closed, and

(b) if $F$ is closed and $F \subset K$ then $F$ is compact.

**Proof. (b)** Let $\mathcal{G}$ be an open cover of $F$. Since $F$ is closed, then $X \setminus F$ is open. So $\mathcal{G} \cup \{X \setminus F\}$ is an open cover of $K$. Since $K$ is compact, there are $G_1, G_2, \ldots, G_n$ in $\mathcal{G}$ such that $K \subset G_1 \cup G_2 \cup \cdots \cup G_n \cup (X \setminus F)$. Since $F \subset K$, then $F \subset G_1 \cup G_2 \cup \cdots \cup G_n$ and so $F$ is compact. □

# Proposition II.4.4

**Proposition II.4.4.** A set $K \subset X$ is compact if and only if every collection $\mathcal{F}$ of closed subsets of $K$ with the finite intersection property satisfies $\cap_{F \in \mathcal{F}} F \neq \varnothing$.

**Proof.** Suppose $K$ is compact and $\mathcal{F}$ is a collection of closed subsets of $K$ having the finite intersection property. ASSUME $\cap_{F \in \mathcal{F}} F = \varnothing$ and let $\mathcal{G} = \cup_{F \in \mathcal{F}} (X \setminus F)$. Then

$$\begin{aligned} \cup_{F \in \mathcal{F}} (X \setminus F) &= X \setminus \cap_{F \in \mathcal{F}} F \text{ by DeMorgan's Laws} \\ &= X \text{ by assumption.} \end{aligned}$$

So $\mathcal{G}$ is an open cover of $K$. Thus, there are $F_1, F_2, \ldots, F_n \in \mathcal{F}$ such that $K \subset \cup_{k=1}^{n} (X \setminus F_k) = X \setminus \cap_{k=1}^{n} F_k$ by DeMorgan. But then $\cap_{k=1}^{n} F_k \subset X \setminus K$.

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**Proposition II.4.4.** A set $K \subset X$ is compact if and only if every collection $\mathcal{F}$ of closed subsets of $K$ with the finite intersection property satisfies $\cap_{F \in \mathcal{F}} F \neq \varnothing$.

**Proof (continued).** Now suppose every collection $\mathcal{F}$ of closed subsets of $K$ with the finite intersection property satisfies $\cap_{F \in \mathcal{F}} F \neq \varnothing$. ASSUME $K$ is not compact. Let $\mathcal{G}$ be an open cover of $K$ with no finite subcover and define $\mathcal{F} = \{K \setminus G \mid G \in \mathcal{G}\}$. Then $\mathcal{F}$ consists of sets closed in $K$ and for any $F_1, F_2, \ldots, F_n \in \mathcal{F}$ we have

$$F_1 \cap F_2 \cap \cdots \cap F_n = (K \setminus G_1) \cap (K \setminus G_2) \cap \cdots \cap (K \setminus G_n) = K \setminus \cup_{k=1}^{n} G_k.$$

Since $K$ is not compact, then $\cup_{k=1}^{n} G_k$ does not cover $K$ and hence $K \setminus \cup_{k=1}^{n} G_k \neq \varnothing$. So $\mathcal{F}$ satisfies the finite intersection property. However, $\cap_{F \in \mathcal{F}} F = K \setminus \cup_{G \in \mathcal{G}} G = \varnothing$ since $\mathcal{G}$ is an open cover of $K$, a CONTRADICTION. So the assumption that $K$ is not compact is false and $K$ is compact. □

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Since $K$ is not compact, then $\cup_{k=1}^n G_k$ does not cover $K$ and hence $K \setminus \cup_{k=1}^n G_k \neq \varnothing$. So $\mathcal{F}$ satisfies the finite intersection property. However, $\cap_{F\in\mathcal{F}} F = K \setminus \cup_{G\in\mathcal{G}} G = \varnothing$ since $\mathcal{G}$ is an open cover of $K$, a CONTRADICTION. So the assumption that $K$ is not compact is false and $K$ is compact. □

# Corollary II.4.5

**Corollary II.4.5.** Every compact metric space is complete.

**Proof.** We use Cantor's Theorem (Theorem II.3.7). Let $\{F_n\}$ be a sequence of non-empty closed sets with $F_1 \supset F_2 \supset \cdots$ and $\text{diam}(F_n) \to 0$. Since the $F$'s are nested, any finite collection satisfies $F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing$ (where $n_1 < n_2 < \cdots n_k$), so $\{F_n\}$ has the finite intersection property.

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# Corollary II.4.6

**Corollary II.4.6.** If $X$ is a compact set in a metric space, then every infinite set has a limit point in $X$.

**Proof.** Let $S$ be an infinite subset of $X$. ASSUME $S$ has no limit points in $X$. Let $\{a_1, a_2, \ldots\}$ be a sequence of distinct points in $S$. Define $F_n = \{a_n, a_{n+1}, \ldots\}$. Then since $S$ has no limit points, set $F_n$ has no limit points for all $n \in \mathbb{N}$.

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# Lemma II.4.8

**Lemma II.4.8. Lebesgue's Covering Lemma.**
If $(X, d)$ is sequentially compact and $\mathcal{G}$ is an open cover of $X$ then there is an $\varepsilon > 0$ such that if $x \in X$, there is a set $G \in \mathcal{G}$ with $B(x; \varepsilon) \subset G$.

**Proof.** Suppose $\mathcal{G}$ is an open cover of $X$. ASSUME there is no such $\varepsilon$. Then for each $n \in \mathbb{N}$ there is a point $x_n \in X$ such that $B(x_n; 1/n)$ is not contained in any set $G \in \mathcal{G}$. Consider the sequence $\{x_n\}$. Since $X$ is sequentially compact, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ for some $x_0 \in X$.

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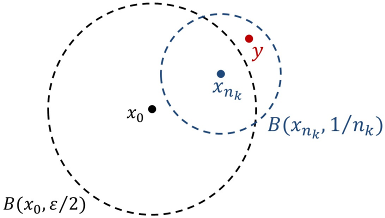
If $(X, d)$ is sequentially compact and $\mathcal{G}$ is an open cover of $X$ then there is an $\varepsilon > 0$ such that if $x \in X$, there is a set $G \in \mathcal{G}$ with $B(x; \varepsilon) \subset G$.

**Proof.** Suppose $\mathcal{G}$ is an open cover of $X$. ASSUME there is no such $\varepsilon$. Then for each $n \in \mathbb{N}$ there is a point $x_n \in X$ such that $B(x_n; 1/n)$ is not contained in any set $G \in \mathcal{G}$. Consider the sequence $\{x_n\}$. Since $X$ is sequentially compact, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ for some $x_0 \in X$. Since $\mathcal{G}$ is a covering of $X$ then $x_0 \in G_0$ for some $G_0 \in \mathcal{G}$ and since $G_0$ is open, for some $\varepsilon > 0$ we have $B(x_0; \varepsilon) \subset G_0$. Let $N \in \mathbb{N}$ be such that $d(x_0; x_{n_k}) < \varepsilon/2$ for all $n_k \geq N$. Next, let $n_k > \max\{N, 2/\varepsilon\}$ and let $y \in B(x_{n_k}; 1/n_k)$ (see the image below).

# Lemma II.4.8 (continued)




Then

$$\begin{aligned} d(x_0, y) &\leq d(x_0, x_{n_k}) + d(x_{n_k}, y) \text{ by the Triangle Inequality} \\ &< \varepsilon/2 + 1/n_k \text{ by the choices of } n_k \text{ and } y \\ &< \varepsilon. \end{aligned}$$

But then $B(x_{n_k}; 1/n_k) \subset B(x_0; \varepsilon) \subset G$. However, we originally chose $x_n$ such that $B(x_n; 1/n)$ is not contained in any $G \in \mathcal{G}$, a CONTRADICTION for $n = n_k$. So the assumption that there is no $\varepsilon > 0$ as described is false, and the result follows. □

# Proposition II.4.9

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(a) $X$ is compact,

(b) every infinite subset of $X$ has a limit point,

(c) $X$ is sequentially compact, and

(d) $X$ is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. This property is called *total boundedness*.

**Proof. (a) implies (b)**: This is Corollary II.4.6.

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**Proof. (a) implies (b)**: This is Corollary II.4.6.

**(b) implies (c)**: Let $\{x_n\}$ be a sequence in $X$. Without loss of generality, the $x_n$ are distinct (if an element is repeated an infinite number of times then it is easy to produce a convergent subsequence; finite repetitions have no effect). By hypothesis, the set $\{x_1, x_2, \ldots\}$ has a limit point, say $x_0 \in X$. So some $x_{n_1} \in B(x_0; 1)$; some $x_{n_2} \in B(x_0, 1/2)$ where $n_2 > n_1$; and in general $x_{n_k} \in B(x_0; 1/k)$ where $n_k > n_{k-1} > \cdots > n_2 > n_1$.

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# Proposition II.4.9 (continued 1)

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**Proof (continued).** Then subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has limit $x_0 \in X$. That is, every sequence has a convergent subsequence and $X$ is sequentially compact.

**(c) implies (d)**: Let $\{x_n\}$ be a Cauchy sequence. By hypothesis, $\{x_n\}$ has a convergent subsequence which converges, say, to $x_0$. Then $\{x_n\} \to x_0$ (by Exercise II.3.8) and so $X$ is complete. Now let $\varepsilon > 0$. Fix $x_1 \in X$. If $X = B(x_1; \varepsilon)$ then we conclude (d) with $n = 1$. Otherwise choose $x_2 \in X \setminus B(x_1; \varepsilon)$. If $X = B(x_1; \varepsilon) \cup B(x_2; \varepsilon)$ then we conclude (d) with $n = 2$. Continue is this way and construct $x_3, x_4, \ldots$. If the process terminates at some $n$, then we conclude (d).

# Proposition II.4.9 (continued 1)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

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**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

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(d) $X$ is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. This property is called *total boundedness*.

**Proof (continued.)** If the process does not terminate then we have a sequence $\{x_n\}$ where for any $n \neq m$ we have $d(x_n, x_m) \geq \varepsilon$. But then sequence $\{x_n\}$ can have no convergent subsequence, violating the hypothesis. So the process must terminate and we have $X = \cup_{k=1}^{m} B(x_k; \varepsilon)$ for some $\{x_1, x_2, \ldots, x_n\}$.

**(d) implies (c)**: Let $\{x_n\}$ be a sequence of distinct points (as in the proof of (b) implies (c) we may assume WLOG that the points are distinct). With $\varepsilon = 1$, we can write $X$ as a finite union of balls of radius 1. Since $\{x_n\}$ is an infinite set, there is some $y_1 \in X$ such that $B(y_1; 1)$ contains an infinite number of the $x_n$'s, say $\{x_n^{(1)}\}_{n=1}^{\infty}$.

# Proposition II.4.9 (continued 2)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(c) $X$ is sequentially compact, and

(d) $X$ is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. This property is called *total boundedness*.

**Proof (continued.)** If the process does not terminate then we have a sequence $\{x_n\}$ where for any $n \neq m$ we have $d(x_n, x_m) \geq \varepsilon$. But then sequence $\{x_n\}$ can have no convergent subsequence, violating the hypothesis. So the process must terminate and we have $X = \cup_{k=1}^{m} B(x_k; \varepsilon)$ for some $\{x_1, x_2, \ldots, x_n\}$.

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# Proposition II.4.9 (continued 3)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(c) $X$ is sequentially compact, and

(d) $X$ is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. This property is called *total boundedness*.

**Proof (continued.)** Similarly with $\varepsilon = 1/2$, there is $y_2 \in X$ and subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ with $\{x_n^{(2)}\} \subset B(y_2; 1/2)$. In general, for each $k \in \mathbb{N}$, $k \geq 2$, there is $y_k \in X$ and subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ with $\{x_n^{(k)}\} \subset B(y_k, 1/k)$. Define $F_k = \{x_n^{(k)}\}^-$. Then $\operatorname{diam}(F_k) \leq 2/k$ and $F_1 \supset F_2 \supset \cdots$. Since $X$ is complete by hypothesis, then by Theorem II.3.7, $\cap_{k=1}^{\infty} F_k = \{x_0\}$ for some $x_0 \in X$. Consider the subsequence of $\{x_n\}$ defined as $\{x_k^{(k)}\}$ (by the recursive way the subsequences are constructed, we are insured that this is in fact a subsequence).

# Proposition II.4.9 (continued 3)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(c) $X$ is sequentially compact, and

(d) $X$ is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. This property is called *total boundedness*.

**Proof (continued.)** Similarly with $\varepsilon = 1/2$, there is $y_2 \in X$ and subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ with $\{x_n^{(2)}\} \subset B(y_2; 1/2)$. In general, for each $k \in \mathbb{N}$, $k \geq 2$, there is $y_k \in X$ and subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ with $\{x_n^{(k)}\} \subset B(y_k, 1/k)$. Define $F_k = \{x_n^{(k)}\}^-$. Then $\text{diam}(F_k) \leq 2/k$ and $F_1 \supset F_2 \supset \cdots$. Since $X$ is complete by hypothesis, then by Theorem II.3.7, $\cap_{k=1}^{\infty} F_k = \{x_0\}$ for some $x_0 \in X$. Consider the subsequence of $\{x_n\}$ defined as $\{x_k^{(k)}\}$ (by the recursive way the subsequences are constructed, we are insured that this is in fact a subsequence).

# Proposition II.4.9 (continued 4)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(a) $X$ is compact,

(c) $X$ is sequentially compact.

**Proof (continued.)** Notice that

$$\begin{aligned} d(x_0, x_k^{(k)}) &\leq \operatorname{diam}(F_k) \text{ (since } x_0, x_k^{(k)} \in F_k) \\ &\leq 2/k \end{aligned}$$

for all $k \in \mathbb{N}$. So $x_k^{(k)} \to x_0 \in X$. So $\{x_n\}$ has a convergent subsequence and $X$ is sequentially compact.

**(c) implies (a)**: Let $\mathcal{G}$ be an open cover of $X$. Since $X$ is hypothesized to be sequentially compact, then by Lebesgue's Covering Lemma (Lemma II.4.8), there is some $\varepsilon > 0$ such that for every $x \in X$, there is a $G \in \mathcal{G}$ where $B(x; \varepsilon) \subset G$.

# Proposition II.4.9 (continued 4)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(a) $X$ is compact,

(c) $X$ is sequentially compact.

**Proof (continued.)** Notice that

$$\begin{aligned} d(x_0, x_k^{(k)}) &\leq \text{diam}(F_k) \text{ (since } x_0, x_k^{(k)} \in F_k) \\ &\leq 2/k \end{aligned}$$

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# Proposition II.4.9 (continued 5)

**Proposition II.4.9.** Let $(X, d)$ be a metric space. The following are equivalent:

(a) $X$ is compact,

(c) $X$ is sequentially compact.

**Proof (continued.)** From above, we have that (c) implies (d), so there are points $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. Again, by Lebesgue's Covering Lemma, for $k = 1, 2, \ldots, n$ we have some $G_k \in \mathcal{G}$ such that $B(x_k; \varepsilon) \subset G_k$. Then $X = \cup_{k=1}^{n} G_k$ and $\{G_1, G_2, \ldots, G_n\}$ is a finite subcover of $\mathcal{G}$. So $X$ is compact and (a) follows. $\square$

# Heine-Borel Theorem

**Theorem II.4.10. Heine-Borel Theorem.**
A subset $K$ of $\mathbb{R}^n$ ($n \geq 1$) is compact if and only if $K$ is closed and bounded.

**Proof.** Suppose $K$ is compact. Then $K$ is closed by Proposition II.4.3(a). By Theorem II.4.9(d), $X$ is totally bounded. So there is $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. We then have that $d(x_1, y) < 2n\varepsilon$ for all $y \in X$ and so $X$ is bounded.

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Now suppose $K$ is closed and bounded. Since $K \subset \mathbb{R}^n$ is bounded, then for some $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we have $K \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = F$. Now $F$ is closed (consider $\mathbb{R}^n \setminus F$) and $\mathbb{R}^n$ is complete (here is where completeness is used—showing $\mathbb{R}^n$ is complete based on the completeness of $\mathbb{R}$ is similar to the proof that $\mathbb{C}$ is complete in Proposition II.3.6). So $F$ is complete by Proposition II.3.8. By Lemma, $F$ is totally bounded. So by Proposition II.4.9(d), set $F$ is compact. Now by Proposition II.4.3(b), since set $K$ is closed, set $K$ is compact. □

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**Proof.** Suppose $K$ is compact. Then $K$ is closed by Proposition II.4.3(a). By Theorem II.4.9(d), $X$ is totally bounded. So there is $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n$ such that $X = \cup_{k=1}^{n} B(x_k; \varepsilon)$. We then have that $d(x_1, y) < 2n\varepsilon$ for all $y \in X$ and so $X$ is bounded.
Now suppose $K$ is closed and bounded. Since $K \subset \mathbb{R}^n$ is bounded, then for some $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we have $K \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = F$. Now $F$ is closed (consider $\mathbb{R}^n \setminus F$) and $\mathbb{R}^n$ is complete (here is where completeness is used—showing $\mathbb{R}^n$ is complete based on the completeness of $\mathbb{R}$ is similar to the proof that $\mathbb{C}$ is complete in Proposition II.3.6). So $F$ is complete by Proposition II.3.8. By Lemma, $F$ is totally bounded. So by Proposition II.4.9(d), set $F$ is compact. Now by Proposition II.4.3(b), since set $K$ is closed, set $K$ is compact. □