

Complex Analysis

Chapter 11. Metric Spaces and the Topology of \mathbb{C}
11.5. Continuity—Proofs of Theorems

Proposition 11.5.3

Proposition 11.5.3. Let $f : (X, d) \rightarrow (\Omega, \rho)$ be a function. The following are equivalent:

- (a) f is continuous (on set X),
- (b) if Δ is open in Ω then $f^{-1}(\Delta)$ is open in X , and
- (c) if Γ is closed in Ω then $f^{-1}(\Gamma)$ is closed in X .

Proof. (a) implies (b): Let Δ be open in Ω and let $x \in f^{-1}(\Delta)$. So $\omega = f(x)$ for some $\omega \in \Delta$. Since Δ is open, there is $\varepsilon > 0$ such that $B(\omega, \varepsilon) \subset \Delta$. Since f is continuous at x by hypothesis, by Proposition 11.5.2(b) there is $\delta > 0$ such that $B(x; \delta) \subset f^{-1}(B(\omega; \varepsilon)) \subset f^{-1}(\Delta)$. So $f^{-1}(\Delta)$ is open.

(b) implies (c): If $\Gamma \subset \Omega$ is closed then $\Delta = \Omega \setminus \Gamma$ is open. By hypothesis $f^{-1}(\Delta) = X \setminus f^{-1}(\Gamma)$ is open (recall that “ $f : X \rightarrow \Omega$ ” means that the domain of f is X). So $f^{-1}(\Gamma)$ is closed.

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Proposition 11.5.3

Proposition 11.5.3 (continued)

Proposition 11.5.3. Let $f : (X, d) \rightarrow (\Omega, \rho)$ be a function. The following are equivalent:

- (a) f is continuous (on set X),
- (c) if Γ is closed in Ω then $f^{-1}(\Gamma)$ is closed in X .

Proof (continued). (c) implies (a): ASSUME there is some $x \in X$ at which f is not continuous. Then by Proposition 11.5.2(c), there is $\varepsilon > 0$ and a sequence $\{x_n\} \subset X$ such that $\rho(f(x_n), f(x)) \geq \varepsilon$ for all $n \in \mathbb{N}$ and yet $x = \lim_{n \rightarrow \infty} x_n$. (We are negating Proposition 11.5.2(c)

here—technically, negating $\alpha = \lim_{n \rightarrow \infty} f(x_n)$ would imply that $\rho(f(x_n), f(x)) \geq \varepsilon$ for some $n \geq N$ and for all $N \in \mathbb{N}$. But this condition allows us to construct a subsequence $\{x_{n_k}\}$ where $\rho(f(x_{n_k}), f(x)) \geq \varepsilon$ for all n_k , so the claim stands “for all $n \in \mathbb{N}$ ” without loss of generality.)

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(b) implies (c): If $\Gamma \subset \Omega$ is closed then $\Delta = \Omega \setminus \Gamma$ is open. By hypothesis $f^{-1}(\Delta) = X \setminus f^{-1}(\Gamma)$ is open (recall that “ $f : X \rightarrow \Omega$ ” means that the domain of f is X). So $f^{-1}(\Gamma)$ is closed.

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Proposition 11.5.3

Proposition 11.5.3 (continued again)

Proposition 11.5.3. Let $f : (X, d) \rightarrow (\Omega, \rho)$ be a function. The following are equivalent:

- (a) f is continuous (on set X),
- (c) if Γ is closed in Ω then $f^{-1}(\Gamma)$ is closed in X .

Proof (continued). Let $\Gamma = \omega \setminus B(f(x); \varepsilon)$. Then Γ is closed (the complement of an open set) and $f(x_n) \in \Gamma$ for all $n \in \mathbb{N}$, so $x_n \in f^{-1}(\Gamma)$ for all $n \in \mathbb{N}$. By hypothesis, $f^{-1}(\Gamma)$ is closed and so contains its limit points by Proposition 11.3.2, so $x = \lim_{n \rightarrow \infty} x_n \in f^{-1}(\Gamma)$. So $f(x) \in \Gamma \setminus B(f(x); \varepsilon)$, a CONTRADICTION. So the assumption that f is not continuous at some $x \in X$ is false. Hence f is continuous on X . \square

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Proposition 11.5.5

Proposition 11.5.5. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous functions. Then $g \circ f : X \rightarrow Z$, where $g \circ f(x) = g(f(x))$, is continuous.

Proof. Let $U \subset Z$ be open. Since g is continuous, then by Proposition 11.5.3(b), $g^{-1}(U) \subset Y$ is open. Since f is continuous, then again by Proposition 11.5.3(b), $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \subset X$ is open. So, by Proposition 11.5.3, $g \circ f$ is continuous on X . \square

Proposition 11.5.7 (continued)

Proposition 11.5.7. Let A be a non-empty subset of X . Then:

- (a) $d(x, A) = d(x, A^-)$,
 (b) $d(x, A) = 0$ if and only if $x \in A^-$.

Proof (continued). In particular, $d(x, a) - d(x, y) < \varepsilon/2$ and $d(x, y) > d(x, a) - \varepsilon/2$. So from (*),
 $d(x, A^-) \geq d(x, y) - \varepsilon/2 > d(x, z) - \varepsilon \geq d(x, A) - \varepsilon$ by the infimum definition and the fact that $a \in A$. Since $\varepsilon > 0$ is arbitrary, we have $d(x, A^-) \geq d(x, A)$. Combining with the above observation, we have $d(x, A^-) = d(x, A)$.

(b) If $x \in A^-$ then $0 = d(x, A^-) = d(x, A)$ by (a). Conversely, for any $x \in X$ there is a "minimizing sequence" $\{a_n\} \subset A$ such that $d(x, A) = \lim_{n \rightarrow \infty} d(x, a_n)$. (This is a property of the infimum definition—for each $n \in \mathbb{N}$, define $\varepsilon \geq 1/n$ and choose $a_n \in A$ such that $d(x, A) + 1/n < d(x, a_n)$. The sequence $\{a_n\}$ is then as desired.) So if $d(x, A) = 0$, then $\lim_{n \rightarrow \infty} d(x, a_n) = 0$. But then $x = \lim_{n \rightarrow \infty} a_n$, and $x \in A^-$. \square

Proposition 11.5.7

Proposition 11.5.7. Let A be a non-empty subset of X . Then:

- (a) $d(x, A) = d(x, A^-)$,
 (b) $d(x, A) = 0$ if and only if $x \in A^-$, and
 (c) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.

Proof. (a) First, for general sets A, B with $A \subset B$ we have $d(X, B) \leq d(x, A)$ by the infimum definition of distance. So $d(x, A^-) \leq d(x, A)$. Now, let $\varepsilon > 0$. By the properties of infimum, there is $y \in A^-$ such that

$$d(x, A^-) \geq d(x, y) - \varepsilon/2. \quad (*)$$

Since $y \in A^-$, there is $a \in A$ such that $d(y, a) < \varepsilon/2$. But by the Triangle Inequality, $d(x, y) \leq d(x, a) + d(a, y)$ and $d(x, a) \leq d(x, y) + d(y, a)$, which imply $d(x, y) - d(x, a) \leq d(a, y) = d(y, a)$ and $-d(y, a) \leq d(x, y) - d(x, a)$, respectively. So $|d(x, y) - d(x, a)| \leq d(y, a) < \varepsilon/2$.

Proposition 11.5.7 (continued)

Proposition 11.5.7. Let A be a non-empty subset of X . Then:

- (c) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.

Proof (continued). (c) For all $a \in A$, $d(x, a) \leq d(x, y) + d(y, a)$ by the Triangle Inequality. Hence

$$\begin{aligned} d(x, A) &= \inf\{d(x, a) \mid a \in A\} \\ &\leq \inf\{d(x, y) + d(y, a) \mid a \in A\} \\ &= d(x, y) + d(y, A). \end{aligned}$$

So $d(x, A) - d(y, a) \leq d(x, y)$. Similarly, interchanging X and Y (and x and y) gives $d(y, A) - d(x, A) \leq d(y, x) = d(x, y)$. So $|d(x, A) - d(y, A)| \leq d(x, y)$, as claimed. \square

Theorem 11.5.8

Theorem 11.5.8. Let $f : (X, d) \rightarrow (\Omega, \rho)$ be a **Continuous function**.

- (a) If X is **Compact**, then $f(X)$ is compact in Ω .
 (b) If X is **Connected**, then $f(X)$ is connected in Ω .

Proof. We may assume WLOG that $f(X) = \Omega$, since consideration of compactness and connectedness are both dealt with in terms of open sets, and if a set U is open relative to $f(X)$ then there is open $O \subset \Omega$ where $U = f(X) \cap O$.

(a) Let $\{\omega_n\}$ be a sequence in Ω . Then there is, for each $n \in \mathbb{N}$, a point $x_n \in X$ with $\omega_n = f(x_n)$. Since X is compact, then it is sequentially compact by Theorem 11.4.9(c) and there is some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x = \lim(x_{n_k})$ for an $x \in X$. Define $\omega \in \Omega$ as $\omega = f(x)$. Since f is continuous, then $\lim f(x_{n_k}) = \lim(\omega_{n_k}) = f(x) = \omega$, and so sequence $\{\omega_n\} \subset \Omega$ has a convergent subsequence $\{\omega_{n_k}\}$. So Ω is sequentially compact, and by Theorem 11.4.9, Ω is compact.

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Theorem 11.5.15

Theorem 11.5.15

Theorem 11.5.15. Suppose $f : X \rightarrow \Omega$ is continuous and X is compact.

Then f is uniformly continuous on X .

Proof. Let $\varepsilon > 0$. ASSUME f is not uniformly continuous on X . Then for all $n \in \mathbb{N}$, with $\delta_n = 1/n$ there are points $x_n, y_n \in X$ with $d(x_n, y_n) < 1/n$ but

$$\rho(f(x_n), f(y_n)) \geq \varepsilon. \quad (*)$$

Since X is compact then X is sequentially compact by Theorem 11.4.9. So for the sequence $\{x_n\}$ there is a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow x$ for some $x \in X$. Now by the Triangle Inequality

$$d(x, y_{n_k}) \leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) < d(x, x_{n_k}) + 1/n_k.$$

As $k \rightarrow \infty$, $d(x, x_{n_k}) \rightarrow 0$ and $1/n_k \rightarrow 0$, so $\lim_{k \rightarrow \infty} d(x, y_{n_k}) = 0$ and $y_{n_k} \rightarrow x$ (this can also be shown with an $\varepsilon/2$ argument). □

Theorem 11.5.8 (continued)

Theorem 11.5.8. Let $f : (X, d) \rightarrow (\Omega, \rho)$ be a **Continuous function**.

- (b) If X is **Connected**, then $f(X)$ is connected in Ω .

Proof. (b) Suppose $\Sigma \subset \Omega$ is both open and closed and $\Sigma \neq \emptyset$. Since $f(X) = \Omega$ (i.e., f is onto Ω), then $f^{-1}(\Sigma) \neq \emptyset$. By Proposition 11.5.3, $f^{-1}(\Sigma)$ is both open and closed. But since X is connected by hypothesis, then $f^{-1}(\Sigma) = X$. But this implies $F(X) = \Sigma = \Omega$, and so $\Sigma = \Omega$ and the only sets in (Ω, ρ) which are both open and closed are \emptyset and Ω . That is, $\Omega = f(X)$ is connected. □

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Theorem 11.5.15

Theorem 11.5.15 (continued)

Theorem 11.5.15. Suppose $f : X \rightarrow \Omega$ is continuous and X is compact.

Then f is uniformly continuous on X .

Proof (continued). But then $f(x) = \lim f(x_{n_k}) = \lim f(y_{n_k})$, so

$$\begin{aligned} \varepsilon &\leq \rho(f(x_{n_k}), f(y_{n_k})) \text{ by } (*) \\ &\leq \rho(f(x_{n_k}), f(x)) + \rho(f(x), f(y_{n_k})) \text{ by the Triangle Inequality.} \end{aligned}$$

But each term of the right hand side can be made arbitrarily small (i.e., less than ε). This **CONTRADICTION** implies that the assumption that f is not uniformly continuous is false, and the result follows. □

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Theorem 11.5.17

Theorem 11.5.17. Let A and B be non-empty disjoint sets in X . If B is closed and A is compact, then $d(A, B) > 0$.

Proof. Define $f : X \rightarrow \mathbb{R}$ by $f(x) = d(x, B)$. Then as commented after Proposition 11.5.7, f is Lipschitz and hence uniformly continuous and continuous on X . Since $A \cap B = \emptyset$ and B is closed, then $f(a) > 0$ for all $a \in A$ since $d(a, B) = 0$ if and only if $a \in B^-$ (by Proposition 11.5.7(b)). But since A is compact, by the Extreme Value Theorem (Corollary 11.5.12) there is some $a \in A$ with $f(a) = \inf\{f(x) \mid x \in A\} = d(A, B)$, and hence $0 < f(a) = d(A, B)$. \square