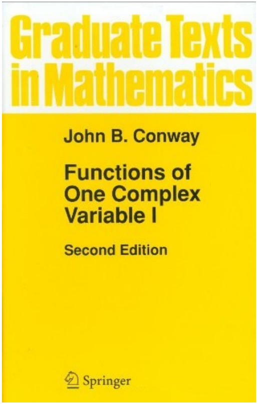
Complex Analysis

Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.5. Continuity—Proofs of Theorems



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Proposition II.5.3. Let $f : (X, d) \to (\Omega, \rho)$ be a function. The following are equivalent:

(a) f is continuous (on set X),

(b) if Δ is open in Ω then $f^{-1}(\Delta)$ is open in X, and

(c) if Γ is closed in Ω then $f^{-1}(\Gamma)$ is closed in X.

Proof. (a) implies (b): Let Δ be open in Ω and let $x \in f^{-1}(\Delta)$. So $\omega = f(x)$ for some $\omega \in \Delta$. Since Δ is open, there is $\varepsilon > 0$ such that $B(\omega, \varepsilon) \subset \Delta$.

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(b) implies (c): If $\Gamma \subset \Omega$ is closed then $\Delta = \Omega \setminus \Gamma$ is open. By hypothesis $f^{-1}(\Delta) = X \setminus f^{-1}(\Gamma)$ is open (recall that "$f : X \to \Omega$" means that the domain of f is X). So $f^{-1}(\Gamma)$ is closed.

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Proof (continued). (c) implies (a): ASSUME there is some $x \in X$ at which f is not continuous. Then by Proposition II.5.2(c), there is $\varepsilon > 0$ and a sequence $\{x_n\} \subset X$ such that $\rho(f(x_n), f(x)) \geq \varepsilon$ for all $n \in \mathbb{N}$ and yet $x = \lim_{n\to\infty} x_n$. (We are negating Proposition II.5.2(c) here—technically, negating $\alpha = \lim_{n\to\infty} f(x_n)$ would imply that $\rho(f(x_n), f(x)) \geq \varepsilon$ for some $n \geq N$ and for all $N \in \mathbb{N}$. But this condition allows us to construct a subsequence $\{x_{n_k}\}$ where $\rho(f(x_{n_k}), f(x)) \geq \varepsilon$ for all n_k, so the claim stands "for all $n \in \mathbb{N}$" without loss of generality.)

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Proposition II.5.5. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$, where $g \circ f(x) = g(f(x))$, is continuous.

Proof. Let $U \subset Z$ be open. Since g is continuous, then by Proposition II.5.3(b), $g^{-1}(U) \subset Y$ is open.

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Proposition II.5.7. Let A be a non-empty subset of X. Then:

(a) $d(x, A) = d(x, A^-)$,

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(c) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.

Proof. **(a)** First, for general sets A, B with $A \subset B$ we have $d(X, B) \leq d(x, A)$ by the infimum definition of distance. So $d(x, A^-) \leq d(x, A)$. Now, let $\varepsilon > 0$.

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$$d(x,A^-) \geq d(x,y) - \varepsilon/2. \qquad (*)$$

Since $y \in A^-$, there is $a \in A$ such that $d(y,a) < \varepsilon/2$.

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So $d(x,A) - d(y,a) \leq d(x,y)$. Similarly, interchanging X and Y (and x and y) gives $d(y,A) - d(x,A) \leq d(y,x) = d(x,y)$. So $|d(x,A) - d(y,A)| \leq d(x,y)$, as claimed. □

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Theorem II.5.8

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(a) If X is **C**ompact, then $f(X)$ is compact in Ω.

(b) If X is **C**onnected, then $f(X)$ is connected in Ω.

Proof. We may assume WLOG that $f(X) = \Omega$, since consideration of compactness and connectedness are both dealt with in terms of open sets, and if a set U is open relative to $f(X)$ then there is open $O \subset \Omega$ where $U = f(X) \cap O$.

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(a) Let $\{\omega_n\}$ be a sequence in Ω. Then there is, for each $n \in \mathbb{N}$, a point $x_n \in X$ with $\omega_n = f(x_n)$. Since X is compact, then it is sequentially compact by Theorem II.4.9(c) and there is some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x = \lim(x_{n_k})$ for an $x \in X$. Define $\omega \in \Omega$ as $\omega = f(x)$.

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(a) If X is **C**ompact, then $f(X)$ is compact in Ω.

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Proof. We may assume WLOG that $f(X) = \Omega$, since consideration of compactness and connectedness are both dealt with in terms of open sets, and if a set U is open relative to $f(X)$ then there is open $O \subset \Omega$ where $U = f(X) \cap O$.

(a) Let $\{\omega_n\}$ be a sequence in Ω. Then there is, for each $n \in \mathbb{N}$, a point $x_n \in X$ with $\omega_n = f(x_n)$. Since X is compact, then it is sequentially compact by Theorem II.4.9(c) and there is some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x = \lim(x_{n_k})$ for an $x \in X$. Define $\omega \in \Omega$ as $\omega = f(x)$. Since f is continuous, then $\lim f(x_{n_k}) = \lim(\omega_{n_k}) = f(x) = \omega$, and so sequence $\{\omega_n\} \subset \Omega$ has a convergent subsequence $\{\omega_{n_k}\}$. So Ω is sequentially compact, and by Theorem II.4.9, Ω is compact.

Theorem II.5.8 (continued)

Theorem II.5.8. Let $f : (X, d) \to (\Omega, \rho)$ be a **C**ontinuous function.

(b) If X is **C**onnected, then $f(X)$ is connected in Ω.

Proof. (b) Suppose $\Sigma \subset \Omega$ is both open and closed and $\Sigma \neq \varnothing$. Since $f(X) = \Omega$ (i.e., f is onto Ω), then $f^{-1}(\Sigma) \neq \varnothing$. By Proposition II.5.3, $f^{-1}(\Sigma)$ is both open and closed. But since X is connected by hypothesis, then $f^{-1}(\Sigma) = X$.

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$$d(x, y_{n_k}) \leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) < d(x, x_{n_k}) + 1/n_k.$$

As $k \to \infty$, $d(x, x_{n_k}) \to 0$ and $1/n_k \to 0$, so $\lim_{k\to\infty} d(x, y_{n_k}) = 0$ and $y_{n_k} \to x$ (this can also be shown with an $\varepsilon/2$ argument).

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But each term of the right hand side can be made arbitrarily small (i.e., less than ε). This CONTRADICTION implies that the assumption that f is not uniformly continuous is false, and the result follows. □

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Theorem II.5.17

Theorem II.5.17. Let A and B be non-empty disjoint sets in X. If B is closed and A is compact, then $d(A, B) > 0$.

Proof. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, B)$. Then as commented after Proposition II.5.7, f is Lipschitz and hence uniformly continuous and continuous on X.

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