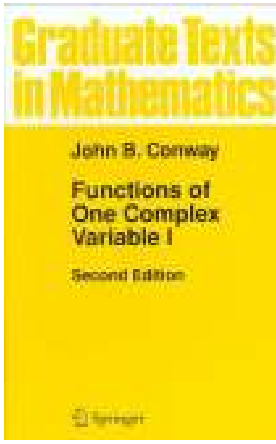
Complex Analysis

Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.6. Uniform Convergence—Proofs of Theorems



Theorem II.6.1

Theorem II.6.1. Suppose $f_n : (X, d) \to (\Omega, \rho)$ is continuous for each $n \in \mathbb{N}$ and suppose $f = \mathrm{u} - \lim(f_n)$. Then f is continuous.

Proof. Let $x_0 \in X$ and let $\varepsilon > 0$. Since $f = \mathrm{u} - \lim(f_n)$, there is a function f_n with

$$\rho(f(x), f_n(x)) < \varepsilon/3 \text{ for all } x \in X. \qquad (*)$$

Since f_n is continuous at x_0 there is $\delta > 0$ such that

$$\rho(f_n(x_0), f_n(x)) < \varepsilon/3 \text{ when } d(x_0, x) < \delta. \qquad (**)$$

So if $d(x_0, x) < \delta$, then we have

$$\begin{aligned} \rho(f(x_0), f(x)) &\leq \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x)) + \rho(f_n(x), f(x)) \\ &\qquad \text{by the Triangle Inequality} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \text{ by } (*),\ (**), \text{ and } (*), \text{ respectively} \\ &= \varepsilon. \end{aligned}$$

So f is continuous at x_0. Since x_0 is arbitrary, f is continuous on X. □

Theorem II.6.2

Theorem II.6.2. Weierstrass *M*-Test.
Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \leq M_n$ for all $x \in X$ and suppose the constants satisfy $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} u_n$ is uniformly convergent.

Proof. Let $f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$. Then for $n > m$

$$|f_n(x) - f_m(x)| = |u_{m+1}(x) + u_{m+2}(x) + \cdots + u_n(x)| \leq \sum_{k=m+1}^{n} M_k$$

for all $x \in X$. Since $\sum_{n=1}^{\infty} M_k$ converges by hypothesis, then it is Cauchy. So $\{f_n(x)\}$ is a Cauchy sequence for each $x \in X$. Since $\mathbb{C}$ is complete, there is $\xi \in \mathbb{C}$ where $\xi = \lim f_n(x)$. Define $f(x) = \xi$ pointwise for each $x \in X$. Then $f : X \to \mathbb{C}$.

Theorem II.6.2 (continued)

Proof (continued). Now

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \sum_{k=n+1}^{\infty} u_k(x) \right| \text{ by definition of } f \\ &\leq \sum_{k=n+1}^{\infty} |u_k(x)| \text{ by the Triangle Inequality and limits} \\ &\leq \sum_{k=n+1}^{\infty} M_k \text{ since } |u_k(x)| \leq M_k \text{ on } X. \end{aligned}$$

Since $\sum_{k=1}^{\infty} M_k$ is convergent, then for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \geq N$ we have $\sum_{k=n+1}^{\infty} M_k < \varepsilon$. So for all $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$ for all $x \in X$. That is, $f = \mathrm{u} - \lim(f_n)$ and so $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on X. □