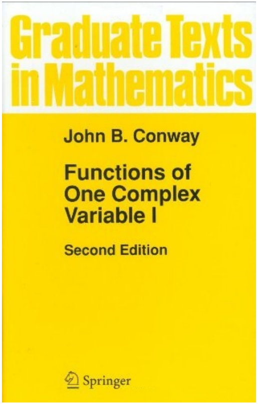
Complex Analysis

Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.6. Uniform Convergence—Proofs of Theorems



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Proof. Let $x_0 \in X$ and let $\varepsilon > 0$. Since $f = \text{u} - \lim(f_n)$, there is a function f_n with

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So if $d(x_0, x) < \delta$, then we have

$$\begin{aligned} \rho(f(x_0), f(x)) &\le \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x)) + \rho(f_n(x), f(x)) \\ &\qquad \text{by the Triangle Inequality} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \text{ by } (*), (**), \text{ and } (*), \text{ respectively} \\ &= \varepsilon. \end{aligned}$$

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Theorem II.6.2. Weierstrass M-Test.
Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \leq M_n$ for all $x \in X$ and suppose the constants satisfy $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} u_n$ is uniformly convergent.

Proof. Let $f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$. Then for $n > m$

$$|f_n(x) - f_m(x)| = |u_{m+1}(x) + u_{m+2}(x) + \cdots + u_n(x)| \leq \sum_{k=m+1}^{n} M_k$$

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$$\begin{aligned} |f(x) - f_n(x)| &= \left| \sum_{k=n+1}^{\infty} u_k(x) \right| \text{ by definition of } f \\ &\leq \sum_{k=n+1}^{\infty} |u_k(x)| \text{ by the Triangle Inequality and limits} \\ &\leq \sum_{k=n+1}^{\infty} M_k \text{ since } |u_k(x)| \leq M_k \text{ on } X. \end{aligned}$$

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