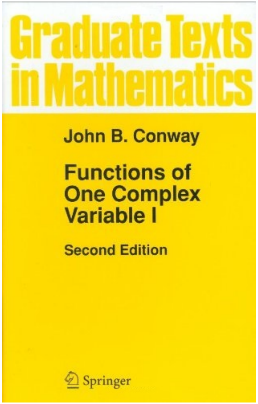
Complex Analysis

Chapter III. Elementary Properties and Examples of Analytic Functions

III.1. Power Series—Proofs of Theorems



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Proof. Let $\varepsilon > 0$. Let z_n be the partial sum of $\sum_{k=1}^{\infty} a_k$: $z_n = a_1 + a_2 + \cdots + a_n$. Since $\sum_{n=1}^{\infty} |a_n|$ converges by hypothesis, then there is $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{N-1} |a_k| - \sum_{n=1}^{\infty} |a_n| \right| = \sum_{n=N}^{\infty} |a_n| < \varepsilon.$$

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So if $m > k \geq N$ then by the Triangle Inequality,

$$|z_m - z_k| = \left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{\infty} |a_n| < \varepsilon,$$

and so $\{z_n\}$ is a Cauchy sequence of complex numbers. Since $\mathbb{C}$ is complete by Proposition II.3.6, then $z_n \to z$ for some $z \in \mathbb{C}$. That is, there is $z \in \mathbb{C}$ with $\sum_{n=1}^{\infty} a_n = z$. □

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Theorem III.1.3

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(a) if $|z-a| < R$, the series converges absolutely,

(b) if $|z-a| > R$, the series diverges, and

(c) if $0 < r < R$ then the series converges uniformly on $|z-a| \le r$. Moreover, R is the only number having properties (a) and (b). R is called the *radius of convergence* of the power series.

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(a) If $|z| < R$, there is r with $|z| < r < R$. Then $1/R < 1/r$ and so there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $|a_n|^{1/n} < 1/r$ (by definition of $\overline{\lim}|a_n|^{1/n}$). So for $n \ge N$, $|a_n| < 1/r^n$ and $|a_n z^n| < (|z|/r)^n$. Next,

$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} |a_n z^n| < \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} \left(\frac{|z|}{r}\right)^n .$$

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Since $|z|/r < 1$, then $\sum_{n=N}^{\infty} a_n z^n$ converges absolutely, and the power series converges absolutely for $|z| < R$.

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Theorem III.1.3. If $\sum_{n=0}^{\infty} a_n(z-a)^n$, define the number R as $\frac{1}{R} = \overline{\lim}\, |a_n|^{1/n}$ (so $0 \le R \le \infty$). Then

(c) if $0 < r < R$ then the series converges uniformly on $|z-a| \le r$.

Proof (continued). (c) Suppose $r < R$ and choose ρ such that $r < \rho < R$. As in the proof of (a), let $N \in \mathbb{N}$ be such that $|a_n| < 1/\rho^n$ for all $n \ge N$. Then if $|z| \le r$, we have $|a_n z^n| < (r/\rho)^n$ for all $n \ge N$, and $(r/\rho) < 1$. Now, the Weierstrass M-Test says (Theorem II.6.2 in Section II.6. Uniform Convergence): "Let $u_n : X \to \mathbb{C}$ be a function from a metric space X to $\mathbb{C}$ such that $|u_n(x)| \le M_n$ for all $x \in X$ and suppose $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} u_n$ is uniformly convergent." So with $M_n = (r/\rho)^n$, we see that the series $\sum_{n=N}^{\infty} u_n = \sum_{n=N}^{\infty} a_n z^n$ converges uniformly on $\{z \mid |z| \le r\}$ (and so does $\sum_{n=0}^{\infty} a_n z^n$), by the Weierstrass M-Test.

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Proof (continued). (b) Let $|z| > R$ and choose r with $|z| > r > R$. Then $1/r < 1/R$. So, by the definition of $\overline{\lim}$, there are infinitely many $n \in \mathbb{N}$ such that $1/r < |a_n|^{1/n}$. For these n, $|a_n z^n| > (|z|/r)^n$ and since $|z|/r > 1$, these terms are unbounded and hence the series diverges for such z by the Test for Divergence (for complex series; for the real case, see online Calculus 2 notes on Section 10.2. Infinite Series, Theorem 7). □

Proposition III.1.4

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Proof. Without loss of generality, $a = 0$. Let $\alpha = \lim |a_n/a_{n+1}|$ and suppose $|z| < r < \alpha$. Then (by the definition of limit of a sequence) there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n/a_{n+1}| > r$.

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Proof (continued). Next, suppose $|z| > r > \alpha$. Then, as above, for some $N \in \mathbb{N}$, for all $n \geq N$ we have $|a_n| < r|a_{n+1}|$. Again, with $B = |a_N r^N|$, for $n \geq N$ we get $|a_n r^n| \geq B = |a_N r^N|$ and $|a_n z^n| \geq B|z|^n/r^n$ which diverges to ∞ as $n \to \infty$ since $|z| > r$. So $a_n z^n \not\to 0$ and by the Test for Divergence (for complex series; for the real case, see online Calculus 2 notes on Section 10.2. Infinite Series, Theorem 7), $\sum_{n=0}^{\infty} a_n z^n$ diverges. Since $r > \alpha$ is arbitrary, then $R \leq \alpha$. Therefore $R = \alpha$. □