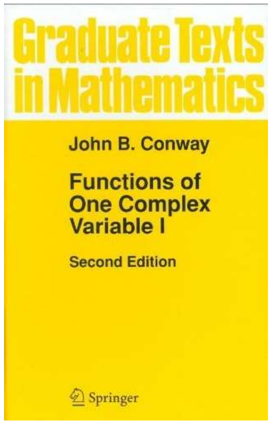
Complex Analysis

Chapter III. Elementary Properties and Examples of Analytic Functions

III.2. Analytic Functions—Proofs of Theorems



Proposition III.2.2

Proposition III.2.2. If $f : G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.

Proof. We have

$$\begin{aligned}\lim_{z\to a}|f(z)-f(a)| &= \lim_{z\to a}\frac{|f(z)-f(a)|}{|z-a|}|z-a| \\ &= \lim_{z\to a}\left|\frac{f(z)-f(a)}{z-a}\right|\lim_{z\to a}|z-a| \text{ since the limit} \\ &\quad \text{of a product is the product of the limits} \\ &\quad \text{(provided the component limits exist)} \\ &= |f'(a)|\cdot 0 = 0.\end{aligned}$$

Therefore $\lim_{z\to a} f(z) = f(a)$ and f is continuous at a, as claimed. □

Chain Rule

Chain Rule. Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and $(g\circ f)'(z) = g'(f(z))f'(z)$ for all $z \in G$.

Proof. Fix $z_0 \in G$ and choose $r > 0$ such that $B(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| < r\} \subset G$. Since $g \circ f$ is continuous at z_0 by Proposition III.2.2 (and properties of continuous functions), it is sufficient to show that if $0 < |h_n| < r$ and $\lim h_n = 0$, then $\lim_{n\to\infty}((g(f(z_0+h_n)) - g(f(z_0))/h_n)$ exists and equals $g'(f(z_0))f'(z_0)$.
Case 1. Suppose $f(z_0) \neq f(z_0 + h_n)$ for all n. Then

$$\frac{g\circ f(z_0+h_n) - g\circ f(z_0)}{h_n} = \frac{g(f(z_0+h_n)) - g(f(z_0))}{f(z_0+h_n)-f(z_0)}\,\frac{f(z_0+h_n)-f(z_0)}{h_n}.$$

By Proposition III.2.2, $\lim_{n\to\infty}(f(z_0+h_n) - f(z_0)) = 0$, so the limit is $g'(f(z_0))f'(z_0)$.

Chain Rule (continued)

Chain Rule. Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and $(g\circ f)'(z) = g'(f(z))f'(z)$ for all $z \in G$.

Proof (continued). Case 2. Suppose $f(z_0) = f(z_0 + h_n)$ for infinitely many n. Then write $\{h_n\}$ as the union of two sequences $\{k_n\}$ and $\{\ell_n\}$ where $f(z_0) \neq f(z_0 + k_n)$ and $f(z_0) = f(z_0 + \ell_n)$ for all n. Since f is differentiable, $f'(z_0) = \lim_{n\to\infty}\frac{f(z_0+\ell_n)-f(z_0)}{\ell_n} = \lim_{n\to\infty}\frac{0}{\ell_n} = 0$. Also, $\lim_{n\to\infty}\frac{g\circ f(z_0+\ell_n)-g\circ f(z_0)}{\ell_n} = \lim_{n\to\infty}\frac{0}{\ell_n} = 0$. By Case 1, $\lim_{n\to\infty}\frac{g\circ f(z_0+k_n)-g\circ f(z_0)}{k_n} = g'(f(z_0))f'(z_0)$. Since $f(z_0) = f(z_0+h_n)$ for infinitely many h_n and f is continuous at z_0, then $f'(z_0) = 0$. Therefore $\lim_{n\to\infty}\frac{g\circ f(z_0+k_n)-g\circ f(z_0)}{h_n} = 0 = g'(f(z_0))f'(z_0)$. Combining Case 1 and case 2, the result follows. □

Proposition III.2.5

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(a) for $k \geq 1$ the series
$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$$
has radius of convergence R,

(b) The function f is infinitely differentiable on $B(a;R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k \geq 1$ and $|z-a| < R$, and

(c) for $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Without loss of generality, assume $a = 0$. **(a)** We prove (a) for $k = 1$ and the result follows in general by induction. By definition, $1/R = \limsup |a_n|^{1/n}$.

Proposition III.2.5 (continued 1)

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(a) for $k \geq 1$ the series $\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$ has radius of convergence R.

Proof (continued). With $k = 1$, we consider $\sum_{n=1}^{\infty} na_nz^{n-1}$ and need to show that $1/R = \limsup |na_n|^{1/(n-1)}$. By L'Hopital's Rule, $\lim_{n\to\infty} n^{1/(n-1)} = 1$. So, by Exercise III.2.2,
$$\limsup |na_n|^{1/(n-1)} = \lim n^{1/(n-1)} \limsup |a_n|^{1/(n-1)} = \limsup |a_n|^{1/(n-1)}.$$
We now need to show that $1/R = \limsup |a_n|^{1/(n-1)}$. Let $1/R' = \limsup |a_n|^{1/(n-1)}$. Then R' is the radius of convergence of
$$\sum_{n=1}^{\infty} a_nz^{n-1} = \sum_{n=0}^{\infty} a_{n+1}z^n.$$

Proposition III.2.5 (continued 2)

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(a) for $k \geq 1$ the series $\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$ has radius of convergence R.

Proof (continued). Now, $z\sum_{n=0}^{\infty} a_{n+1}z^n + a_0 = \sum_{n=0}^{\infty} a_nz^n$ and so if $|z| < R'$ then $\sum_{n=0}^{\infty} |a_nz^n| = |a_0| + |z|\sum_{n=0}^{\infty} |a_{n+1}z^n| < \infty$. So $R' \leq R$. If $|z| < R$ and $z \neq 0$ then $\sum_{n=0}^{\infty} |a_nz^n| < \infty$ and
$$\sum_{n=0}^{\infty} |a_{n+1}z^n| = \frac{1}{|z|}\sum_{n=0}^{\infty} |a_nz^n| - \frac{1}{|z|}|a_0| < \infty$$
and so $R \leq R'$. Therefore $R = R'$. $\square$

Proposition III.2.5 (continued 3)

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(b) The function f is infinitely differentiable on $B(a;R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k \geq 1$ and $|z-a| < R$.

Proof. (b) For $|z| < R$ put $g(z) = \sum_{n=1}^{\infty} na_nz^{n-1}$, $s_n(z) = \sum_{k=0}^{n} a_kz^k$ and $R_n(z) = \sum_{k=n+1}^{\infty} a_kz^k$ (so $f(z) = s_n(z) + R_n(z)$). Fix $w \in B(0;R) = \{z \mid |z-0| < R\}$ and fix r with $|w| < r < R$. We will show $f'(w) = g(w)$. Let $\delta_1 > 0$ be such that $\overline{B}(w,\delta_1) = \{z \mid |w-z| \leq \delta_1\} \subset B(0;r) = \{z \mid |z-0| < r\}$. Let $z \in B(w;\delta_1)$. Then
$$\frac{f(z)-f(w)}{z-w} - g(w) = \left(\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right) + (s_n'(w) - g(w)) + \left(\frac{R_n(z)-R_n(w)}{z-w}\right). \quad (2.8)$$

Proposition III.2.5 (continued 4)

Proposition III.2.5. Let $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R>0$. Then:

(b) The function f is infinitely differentiable on $B(a;R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k\geq 1$ and $|z-a|<R$.

Proof (continued). (b) Now

$$\frac{R_n(z)-R_n(w)}{z-w}=\frac{1}{z-w}\left(\sum_{k=n+1}^{\infty} a_k(z^k-w^k)\right)=\sum_{k=n+1}^{\infty} a_k\left(\frac{z^k-w^k}{z-w}\right).$$

But $\dfrac{|z^k-w^k|}{|z-w|}=|z^{k-1}+z^{k-2}w+\cdots+zw^{k-2}+w^{k-1}|\leq kr^{k-1}$ (since $w,z<r$). Hence,

$$\left|\frac{R_n(z)-R_n(w)}{z-w}\right|\leq\sum_{k=n+1}^{\infty}|a_k|kr^{k-1}.$$

Proposition III.2.5 (continued 5)

Proposition III.2.5. Let $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R>0$. Then:

(b) The function f is infinitely differentiable on $B(a;R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k\geq 1$ and $|z-a|<R$.

Proof (continued). (b) Since $r<R$, then $\sum_{n=1}^{\infty}|a_k|kr^{k-1}$ converges (consider part (a) with $|z|=r<R$). So for any $\varepsilon>0$ there is $N_1\in\mathbb{N}$ such that for all $n\geq N_1$, we have $\left|\frac{R_n(z)-R_n(w)}{z-w}\right|<\frac{\varepsilon}{3}$ (here, $z\in B(w;\delta_1)$). By the definitions of s_n and g, $\lim_{n\to\infty}s_n'(w)=g(w)$, so there exists $N_2\in\mathbb{N}$ such that for all $n\geq N_2$ we have $|s_n'(w)-g(w)|<\varepsilon/3$. Let $n=\max\{N_1,N_2\}$. Then there is $\delta_2>0$ such that $\left|\frac{s_n(z)-s_n(w)}{z-w}-s_n'(w)\right|<\frac{\varepsilon}{3}$ whenever $0<|z-w|<\delta_2$. With $z\in B(w;\delta)$, $z\neq w$, where $\delta=\min\{\delta_1,\delta_2\}$ we have from (2.8) that $\left|\frac{f(z)-f(w)}{z-w}\right|<\varepsilon$. That is, $f'(w)=g(w)$. □

Proposition III.2.5 (continued 6)

Proposition III.2.5. Let $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R>0$. Then:

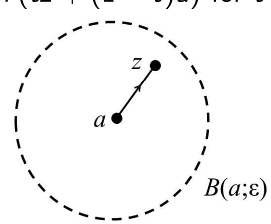
(c) For $n\geq 0$, $a_n=\dfrac{1}{n!}f^{(n)}(a)$.

Proof. From part (a), we have $f^{(k)}(0)=k!a_k$ and the result follows. □

Proposition III.2.10.

Proposition III.2.10. If G is open and connected and $f:G\to\mathbb{C}$ is differentiable with $f'(z)=0$ for all $a\in\mathbb{C}$, then f is constant.

Proof. Fix $z_0\in G$ and denote $\omega_0=f(z_0)$. Let $A=\{z\in G\mid f(z)=\omega_0\}$. Let $z\in G$ and $\{z_n\}\subset A$ where $z=\lim z_n$. Since $f(z_n)=\omega_0$ for $n\in\mathbb{N}$ (each $z_n\in A$) and f is continuous (since f is differentiable; Proposition III.2.2) then $f(z)=f(\lim z_n)=\lim f(z_n)=\omega_0$, and so $z\in A$. So A contains all of its limit points and by Proposition II.3.4 A is closed in G. Next, fix $a\in A$ and let $\varepsilon>0$ be such that $B(a;\varepsilon)\subset G$ (since G is open). If $x\in B(a;\varepsilon)$, set $g(t)=f(tz+(1-t)a)$ for $t\in[0,1]$:



Proposition III.2.10 (continued).

Proposition III.2.10 (continued). If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $a \in \mathbb{C}$, then f is constant.

Proof (continued). Now $g : [0,1] \to \mathbb{C}$ is a composition of differentiable $f : G \to \mathbb{C}$ with $h : [0,1] \to G$ where $h(t) = tz + (1-t)a$. To differentiate g, we need a version of the Chain Rule which is applicable to this setting—this is given in Appendix A on page 304 in Proposition A.4. So

$$g'(t) = \lim_{t\to s} \frac{g(t) - g(s)}{t - s} = f'(tz + (1-t)a)(z - a) = 0$$

since $f'(z) = 0$ for $z \in G$. So $g'(t) = 0$ for $t \in [0,1]$ and hence (by Proposition A.3 of Appendix A on page 303, as applied to $g : [0,1] \to \mathbb{C}$) we have that g is constant. Therefore, $f(z) = g(1) = g(0) = f(a) = \omega_0$. Hence $B(a;\varepsilon) \subset A$ and A is open in G.
Since A is both open and closed in G, $a \neq \varnothing$ (since $a \in A$), and G is connected, then $A = G$. That is, f is constant on G. □

Lemma III.2.A

Lemma III.2.A. Properties of e^z include:

(a) $e^{a+b} = e^a e^b$,
(b) $e^z \neq 0$ for all $z \in \mathbb{C}$,
(c) $\overline{e^z} = e^{\overline{z}}$, and
(d) $|e^z| = e^{\text{Re}(z)}$.

Proof.
(a) Define $g(z) = e^z e^{a+b-z}$ for given $a, b \in \mathbb{C}$. Then $g'(z) = e^z e^{a+b-z} + e^z(-e^{a+b-z}) = 0$. So by Proposition 2.10, $g(z)$ is constant for all $z \in \mathbb{C}$. With $z = 0$, we have $g(0) = e^0 e^{a+b} = e^{a+b}$, so $e^z e^{a+b-z} = e^{a+b}$ for all $z \in \mathbb{C}$. With $z = b$ we have $e^b e^a = e^{a+b}$, as claimed.

(b) By part (a) we have $1 = e^0 = e^z e^{-z}$ for all $z \in \mathbb{C}$, and so $e^z \neq 0$ for all $z \in \mathbb{C}$, as claimed.

Lemma III.2.A (continued 1)

Lemma III.2.A. Properties of e^z include:

(c) $\overline{e^z} = e^{\overline{z}}$, and
(d) $|e^z| = e^{\text{Re}(z)}$.

Proof (continued).
(c) Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, then

$$\begin{aligned} \overline{e^z} &= \overline{\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)} = \overline{\left(\lim_{N\to\infty} \sum_{n=0}^{N} \frac{z^n}{n!}\right)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \overline{\left(\frac{z^n}{n!}\right)} \text{ since conjugation is continuous,} \\ &\quad \text{and Theorem I.2.A} \\ &= \sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!} = e^{\overline{z}}, \text{ as claimed.} \end{aligned}$$

Lemma III.2.A (continued 2)

Lemma III.2.A. Properties of e^z include:

(a) $e^{a+b} = e^a e^b$,
(b) $e^z \neq 0$ for all $z \in \mathbb{C}$,
(c) $\overline{e^z} = e^{\overline{z}}$, and
(d) $|e^z| = e^{\text{Re}(z)}$.

Proof (continued).
(d) By (c) we have

$$|e^z|^2 = e^z\overline{e^z} = e^z e^{\overline{z}} = e^{z+\overline{z}} \text{ by (a)}$$

and so $|e^z|^2 = e^{2\text{Re}(z)}$ and $|e^z| = e^{\text{Re}(z)}$, as claimed. □

Proposition III.2.19

Proposition III.2.19. If $G \subseteq \mathbb{C}$ is open and connected and f is a branch of $\log z$ on G, then the totality of branches of $\log z$ are the functions $\{f(z) + 2k\pi i \mid k \in \mathbb{Z}\}$.

Proof. First, if $g(z) = f(z) + 2k\pi i$, then

$$\exp(g(z)) = \exp(f(z) + 2k\pi i) = \exp(f(z))\exp(2k\pi i) = \exp(f(z)) = z,$$

so g is a branch of $\log z$. Secondly, if $z \in G$ and f and g are both branches of $\log z$, then $\exp(f(z) - g(z)) = \exp(f(z))/\exp(g(z)) = z/z = 1$ and so $f(z) - g(z) = 2k\pi i$ for some $k \in \mathbb{Z}$. Notice that by defining $h(z) = \frac{1}{2\pi i}(f(z) - g(z))$, we now have that $h(z) \subset \mathbb{Z}$ (it's the "k" above). Since h is continuous and G is connected, then $h(G)$ is a connected subset of $\mathbb{Z}$. Therefore $h(G) = \{k\}$ for some fixed $k \in \mathbb{Z}$ and the same k "works" for each $z \in G$. □

Proposition III.2.20

Proposition III.2.20. Let G and Ω be open subsets of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous where $f(G) \subseteq \Omega$ and $g(f(z)) = z$ for all $z \in G$. If g is differentiable and $g'(z) = 0$, then f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$. If g is analytic, then f is analytic.

Proof. Fix $a \in G$ and let $h \in \mathbb{C}$ such that $h \neq 0$ and $a + h \in G$. Then $a = g(f(a))$ (since $g(f(z)) = z$) and $a + h = g(f(a+h))$ implies $g(f(a)) \neq g(f(a+h))$ and so $f(a) \neq f(a+h)$ (or else these two *would* be equal since it would be g evaluated at the same point). So

$$1 = \frac{(a+h) - (a)}{h} = \frac{g(f(a+h)) - g(f(a))}{h}$$

$$= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \frac{f(a+h) - f(a)}{h}.$$

Proposition III.2.20 (continued)

Proposition III.2.20. Let G and Ω be open subsets of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous where $f(G) \subseteq \Omega$ and $g(f(z)) = z$ for all $z \in G$. If g is differentiable and $g'(z) = 0$, then f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$. If g is analytic, then f is analytic.

Proof (continued). The limit is of course 1, and since $\lim_{h\to 0}(f(a+h) - f(a)) = 0$, then

$$\lim_{h\to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} = g'(f(a)) \neq 0.$$

Hence $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ exists and $f'(a) = 1/g'(f(a))$. So $f'(z) = 1/g'(f(z))$ for $z \in G$. If g is analytic, then g' is continuous, and $g'(f(z))$ is continuous. Therefore, f is analytic. □

Theorem III.2.29

Theorem III.2.29. Let u and v be real-valued functions defined on a region G and suppose u and v have continuous partial derivatives (so we view u and v as functions of x and y where $z = x + iy$). Then $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if the Cauchy-Riemann equations are satisfied.

Proof. (Analytic implies Cauchy-Riemann) Let $f : G \to \mathbb{C}$ be analytic and for $z = x + iy \in G$, $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. We know $f'(z) = \lim_{h\to 0} \frac{f(x+h)-f(z)}{h}$ exists. We consider the limit along two paths. We have for $h \in \mathbb{R}$

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+h+iy) - f(x+iy)}{h}$$

$$= \left(\frac{u(x+h,y) - u(x,y)}{h}\right) + i\left(\frac{v(x+h,y) - v(x,y)}{h}\right)$$

and when $h \in \mathbb{R}$ and $h \to 0$ we see that...

Theorem III.2.29 (continued 1)

Proof (continued).

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

Next, let $h \in \mathbb{R}$ and $ih \to 0$. Then

$$\frac{f(z+ih)-f(z)}{ih} = \left(\frac{u(x,y+h)-u(x,y)}{ih}\right) + i\left(\frac{v(x,y+h)-v(x,y)}{ih}\right)$$

$$= -i\left(\frac{u(x,y+h)-u(x,y)}{h}\right) + \left(\frac{v(x,y+h)-v(x,y)}{h}\right)$$

and so $f'(z) = -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$. Therefore, $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ and $\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$. That is, the Cauchy-Riemann equations are necessary. (We have only used *differentiability* here!)

Theorem III.2.29 (continued 2)

Proof (continued). (Cauchy-Riemann implies analytic) Let G be a region and let u and v be functions defined on G with continuous partial derivatives which satisfy the Cauchy-Riemann equations on G. Let $z = x+iy \in G$ and let $B(z;r) \subset G$. If $h = s+it \in B(0;r)$ then

$$\mathrm{Re}(f(z+h)-f(z)) = u(x+s,y+t) - u(x,y)$$

$$= [u(x+s,y+t) - u(x,y+t)] + [u(x,y+t) - u(x,y)].$$

Treating the first bracketed quantity as a function of the first variable and the second bracketed quantity as a function of the second variable, we have by the Mean Value Theorem that for some s_1, t_1 where s_1 is between 0 and s and t_1 is between 0 and t:

$$u_x(x+s_1,y+t) = \frac{u(x+s,y+t)-u(x,y+t)}{s-0},$$

$$u_y(x,y+t_1) = \frac{u(x,y+t)-u(x,y)}{t-0}.$$

Theorem III.2.29 (continued 3)

Proof (continued). Define

$$\varphi(s,t) = [u(x+s,y+t)-u(x,y)] - [u_x(x,y)s + u_y(x,y)t] \quad (*)$$

and then

$$\frac{\varphi(s,t)}{s+it} = \frac{s}{s+it}[u_x(x+s_1,y+t)-u_x(x,y)] + \frac{t}{s+it}[u_y(x,y+t_1)-u_y(x,y)].$$

Since $|s| \le |s+it|$ and $|t| \le |s+it|$, then $\left|\frac{s}{s+it}\right|$ and $\left|\frac{t}{s+it}\right|$ are bounded. Since $|s_1| < |s|$ and $|t_1| < |t|$, and the fact that u_x and u_y are continuous [the continuity of the partials is used here!], then $\lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0$. So by $(*)$,

$$u(x+s,y+t) - u(x,y) = u_x(x,y)s + u_y(x,y)t + \varphi(s,t)$$
$$\text{where } \lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0. \quad (**)$$

Theorem III.2.29 (continued 4)

Proof (continued). Similarly

$$v(x+s,y+t) - v(x,y) = v_x(x,y)s + v_y(x,y)t + \psi(s,t)$$
$$\text{where } \lim_{s+it\to 0} \frac{\psi(s,t)}{s+it} = 0. \quad (***)$$

Now

$$\frac{f(z+s+it)-f(z)}{s+it} = \frac{\mathrm{Re}(f(z+s+it)-f(z)) + i\mathrm{Im}(f(z+s+it)-f(z))}{s+it}$$

$$= \frac{u(x+s,y+t)-u(x,y) + i(v(x+s,y+t)-v(x,y))}{s+it}$$

$$= \frac{(u_x(x,y)s + u_y(x,y)t + \varphi(s,t)) + i(v_x(x,y)s + v_y(x,y)t + \psi(s,t))}{s+it}$$

by $(**)$ and $(***)$

$$= \frac{(u_x(x,y)s - v_x(x,y)t) + i(v_x(x,y)s + u_x(x,y)t)}{s+it} + \frac{\varphi(s,t) + i\psi(s,t)}{s+it}$$

by the Cauchy-Riemann equations

Theorem III.2.29 (continued 5)

Proof (continued).

$$= u_x(x,y) + \frac{i^2 v_x(x,y)t + iv_x(x,y)s}{s+it} + \frac{\varphi(s,t) + i\psi(s,t)}{s+it}$$

$$= u_x(x,y) + iv_x(x,y) + \frac{\varphi(s,t) + i\psi(s,t)}{s+it}.$$

With $s + it \to 0$ and since f is differentiable,

$$f'(z) = u_x(x,y) + iv_x(x,y).$$

Since u_x and v_x are continuous, then f' is continuous and so f is analytic (i.e., continuously differentiable). □